

The determinant formulation of
discrete and continuous two-point
boundary value problems.

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Abstract

In this report we show that discretizations of the integral equation formulation of the two-point boundary value problem

$$\begin{cases} (py')' + qy = f(t,y), \\ a_{11}y(0) - a_{12}y'(0) = c, \quad b_{21}y(1) + b_{22}y'(1) = d, \end{cases}$$

after a simple transformation, called the determinant formulation, have the desirable properties that they are tridiagonal and in general of "positive type". Relations between Peano's kernel theorem, the determinant formulation, and the variational formulation of boundary value problems are studied. We also show that discrete boundary value problems have a determinant formulation.

1. INTRODUCTION

In this report we consider the boundary value problem

$$(1.1) \quad \begin{cases} Ly = (py')' + qy = f(t, y), & 0 < t < 1, \\ B_1 y = a_{11}y(0) - a_{12}y'(0) = c, \\ B_2 y = b_{21}y(1) + b_{22}y'(1) = d. \end{cases}$$

It can also be formulated as an equivalent integral equation

$$(1.2) \quad y(x) = \int_0^1 G(x, t) f(t, y(t)) dt + \frac{cb(x)}{B_1(b)} + \frac{da(x)}{B_2(a)}, \quad 0 \leq x \leq 1,$$

where

$$G(x, t) = \begin{cases} b(x)a(t), & x \geq t, \\ a(x)b(t), & x \leq t, \end{cases}$$

is Green's function for L , subject to the boundary conditions B_1 and B_2 of (1.1).

The integral equation formulation of two-point boundary value problems has not been a popular tool for the numerical solution of these problems. The approximation of the integral equation (1.2) by a quadrature formula, say the trapezoidal rule, gives a system with a dense matrix, while the finite-difference approximation of the differential equation (1.1) gives a tridiagonal matrix. However, in chapter 3 we will show that the integral equation can be written in a form which gives a tridiagonal matrix. We will call this form the determinant formulation of the boundary value problem (or of the integral equation). In chapter 7 we will show that the determinant formulation is equivalent to the variational formulation of the boundary value problem (and hence also to the integral equation formulation) on a given grid. For example the finite-difference

approximation

$$p\left(\frac{i+\frac{1}{2}}{n}\right) \left[y\left(\frac{i+1}{n}\right) - y\left(\frac{i}{n}\right) \right] - p\left(\frac{i-\frac{1}{2}}{n}\right) \left[y\left(\frac{i}{n}\right) - y\left(\frac{i-1}{n}\right) \right] = h^2 f\left(\frac{i}{n}, y\left(\frac{i}{n}\right)\right)$$

of the differential equation (1.1) is also the approximation of the integral equation, the determinant formulation, and the variational formulation by the trapezoidal rule and the midpoint rule.

In chapter 2 we will derive the integral equation formulation of two-point boundary value problems.

In chapter 3 we will derive the determinant formulation of boundary value problems and we will study this formulation analytic.

In chapter 4 we consider the discretization of the determinant formulation. We will give both order h^2 and order h^4 methods.

In chapter 5 we obtain global error estimates and existence theorems for the discrete equations. We also show that the matrices of the determinant formulation are "positive".

In chapter 6 we will consider the relation between the determinant formulation and Peano's kernel theorem. We also give a proof of Peano's kernel theorem based on the theory of distributions. In chapter 7 we will consider the relation between the determinant formulation and the variational formulation of boundary value problems. In chapter 8 we will consider the determinant formulation of discrete boundary value problems, i.e. problems with a tridiagonal matrix.

The determinant formulation of two-point boundary value problems was first used in Aulin [2], [3], [5] to construct a class of Newton-like methods for the numerical solution of these problems. Here the property that the determinant formulation can easily be

shifted from one grid to another is very useful. The application of the determinant formulation to the construction of Newton-like methods for boundary value problems will also appear in Aulin [6]. In the appendix of this report we will define the Newton-like methods for completeness.

2. INTEGRAL EQUATION FORMULATION OF TWO-POINT BOUNDARY VALUE PROBLEMS

Consider the two-point boundary value problem

$$(2.1) \quad \begin{cases} Ly = (py')' + qy = f(t, y(t)), & 0 < t < 1, \\ B_1 y = a_{11}y(0) - a_{12}y'(0) = c, \\ B_2 y = b_{21}y(1) + b_{22}y'(1) = d, \end{cases}$$

where $a_{11}, a_{12}, b_{21}, b_{22} \geq 0$, $a_{11} + b_{21} > 0$, $a_{11} + a_{12} > 0$, and $b_{21} + b_{22} > 0$; p', p, q, f are continuous and $p > 0$ on $[0, 1]$.

Let $\mathcal{B} = \{y \in C^2[0, 1] \mid B_1 y = B_2 y = 0\}$. We assume that zero is not an eigenvalue of $(L, \mathcal{B})$. Then Green's function exists and is unique (see Stakgold [23] p. 66). To construct Green's function for $(L, \mathcal{B})$ choose $a, b \in C^2$ with

$$(2.2) \quad \begin{cases} La = 0, & B_1 a = 0, & a \neq 0, \text{ e.g. } a(0) = a_{12}, & a'(0) = a_{11}, \\ Lb = 0, & B_2 b = 0, & b \neq 0, \text{ e.g. } b(1) = -b_{22}, & b'(1) = b_{21}. \end{cases}$$

Then Green's function for $(L, \mathcal{B})$ is

$$(2.3) \quad G(x, t) = \begin{cases} b(x)a(t)/C, & x \geq t, \\ a(x)b(t)/C, & x \leq t, \end{cases}$$

where C is the constant $p(ab' - a'b) \neq 0$. In the sequel set $a := a/C$, then the Wronskian of a and b is

$$(2.4) \quad ab' - a'b = 1/p.$$

The integral equation formulation of (2.1) is

$$(2.5) \quad y(x) = \int_0^1 G(x, t)f(t, y(t))dt + cb(x)/B_1(b) + da(x)/B_2(a),$$

for $0 \leq x \leq 1$, where $B_2(a) \neq 0$ and $B_1(b) \neq 0$ since zero is not an eigenvalue of $(L, \mathcal{B})$ (see Stakgold [23] pp. 64-79).

If $a_{12} > 0$ then $a(0) \neq 0$ and by (2.4) we get

$$\begin{aligned} \frac{1}{B_1(b)} &= \frac{1}{a_{11}b(0) - a_{12}b'(0)} = \frac{a(0)}{a_{11}a(0)b(0) - a_{12}a(0)b'(0)} = \\ &= \frac{a(0)}{a_{11}a(0)b(0) - a_{12}(1/p(0) + a'(0)b(0))} = \\ &= \frac{a(0)}{b(0)B_1(a) - a_{12}/p(0)}. \end{aligned}$$

But $B_1(a) = 0$ and we get

$$(2.6) \quad \frac{1}{B_1(b)} = - \frac{a(0)p(0)}{a_{12}}.$$

If $b_{22} > 0$ we have in the same way

$$(2.7) \quad \frac{1}{B_2(a)} = - \frac{b(1)p(1)}{b_{22}}.$$

In general Green's function for $(L, \mathcal{B})$ is not known. However, if we write the differential equation in (2.1) as

$$L_p y = (py')' = f(t, y(t)) - q(t)y(t)$$

then Green's function for $(L_p, \mathcal{B})$ is easy to obtain. We will now consider the special boundary value problem

$$(2.8) \quad \begin{cases} L_p y = (py')' = f(t, y(t)), & 0 < t < 1, \\ B_1 y = a_{11}y(0) - a_{12}y'(0) = c, \\ B_2 y = b_{21}y(1) + b_{22}y'(1) = d, \end{cases}$$

where p , p' , and f are continuous and $p > 0$ on the interval $[0, 1]$, and a_{11} , a_{12} , b_{21} , $b_{22} \geq 0$, $a_{11} + b_{21} > 0$, $a_{11} + a_{12} > 0$, $b_{21} + b_{22} > 0$. Let $\mathcal{B} = \{ y \in C^2[0, 1] \mid B_1 y = B_2 y = 0 \}$.

Theorem 2.1

All eigenvalues λ of the eigenvalue problem

$$(py')' + \lambda y = 0$$

$$B_1 y = B_2 y = 0.$$

are greater than zero.

Proof: Assume that y is an eigenfunction belonging to the eigenvalue λ with $\|y\|_2^2 = \int_0^1 y(t)^2 dt = 1$. A routine integration by parts gives

$$\begin{aligned} \lambda = \lambda \|y\|_2^2 &= -p(1)y(1)y'(1) + p(0)y(0)y'(0) + \int_0^1 p(t)y'(t)^2 dt = \\ &= p(1)Bb_{21}y(1)^2 + p(0)Aa_{11}y(0)^2 + \int_0^1 p(t)y'(t)^2 dt \geq 0 \end{aligned}$$

where

$$A = \begin{cases} 1/a_{12} & \text{if } a_{12} > 0, \\ 0 & \text{if } a_{12} = 0, \end{cases} \quad B = \begin{cases} 1/b_{22} & \text{if } b_{22} > 0, \\ 0 & \text{if } b_{22} = 0. \end{cases}$$

If $a_{12} + b_{22} > 0$ we have $\lambda > 0$. If $a_{12} = b_{22} = 0$ and $\lambda = 0$ we must have y constant. But y can only be constant if $a_{11} = b_{21} = 0$ which contradict the condition $a_{11} + b_{21} > 0$.

REMARK.

We note that zero is not an eigenvalue of $(L_p, \mathcal{B})$.

Put

$$(2.9) \quad P(x) = \int_0^x \frac{dt}{p(t)}, \quad Q(x) \equiv 1, \quad 0 \leq x \leq 1.$$

Then P and Q are two linearly independent solutions of $L_p y = 0$.

By (2.2) we define

$$(2.10) \quad \begin{cases} \alpha(x) = a_{11}p(0)P(x) + a_{12}, \\ b(x) = b_{21}p(1)[P(x) - P(1)] - b_{22}. \end{cases}$$

Thus $L_p \alpha = B_1 \alpha = 0$ and $L_p b = B_2 b = 0$. We also have

$$(2.11) \quad p(\alpha b' - \alpha' b) = \\ = a_{11}b_{21}p(0)p(1)P(1) + a_{11}b_{22}p(0) + a_{12}b_{21}p(1) = D_p > 0.$$

Hence with

$$(2.12) \quad a(x) = \alpha(x)/D_p$$

the integral equation formulation of the boundary value problem (2.8) is

$$(2.13) \quad y(x) = \int_0^1 G(x,t)f(t,y(t))dt + \frac{cb(x)}{B_1(b)} + \frac{da(x)}{B_2(a)}, \quad 0 \leq x \leq 1,$$

where

$$(2.14) \quad G(x,t) = \begin{cases} b(x)a(t), & x \geq t, \\ a(x)b(t), & x \leq t, \end{cases}$$

and a, b are defined by (2.10), (2.11), and (2.12).

Another important differential operator for which Green's function is known is $Ly = y'' - k^2 y$. Consider the boundary value problem

$$(2.15) \quad \begin{cases} Ly = y'' - k^2 y = f(t, y(t)), & 0 < t < 1, \\ B_1 y = a_{11}y(0) - a_{12}y'(0) = c, \\ B_2 y = b_{21}y(1) + b_{22}y'(1) = d, \end{cases}$$

where f is continuous, $a_{11}, a_{12}, b_{21}, b_{22} \geq 0$, $a_{11} + b_{21} > 0$, $a_{11} + a_{12} > 0$, $b_{21} + b_{22} > 0$. Put

$$(2.16) \quad \begin{cases} \alpha(x) = \frac{a_{11}}{k} \sinh(kx) + a_{12} \cosh(kx) \\ b(x) = \frac{b_{21}}{k} \sinh(k(x-1)) - b_{22} \cosh(k(x-1)), \end{cases}$$

where we for simplicity assume that $k \geq 0$. Then we have

$$\begin{cases} L\alpha = 0 \\ B_1\alpha = 0 \end{cases} \quad \begin{cases} Lb = 0 \\ B_2b = 0. \end{cases}$$

We also have

$$(2.17) \quad \alpha b' - \alpha' b = \frac{a_{11} b_{21}}{k} \sinh(k) + a_{11} b_{22} \cosh(k) + \\ + a_{12} b_{21} \cosh(k) + k a_{12} b_{22} \sinh(k) = D_k > 0.$$

We note that

$$(2.18) \quad D_k = B_2(\alpha) = -B_1(b).$$

Put

$$(2.19) \quad a(x) = \alpha(x)/D_k.$$

Then the integral equation formulation of the boundary value problem (2.15) is

$$(2.20) \quad y(x) = \int_0^1 G(x,t) f(t, y(t)) dt + \frac{cb(x)}{B_1(b)} + \frac{da(x)}{B_2(a)}, \quad 0 \leq x \leq 1,$$

where

$$(2.21) \quad G(x,t) = \begin{cases} b(x)a(t), & x \geq t, \\ a(x)b(t), & x \leq t, \end{cases}$$

and a, b are defined by (2.16), (2.17), and (2.19).

If $k = 0$ we get

$$(2.22) \quad \begin{cases} D_0 = a_{11} b_{21} + a_{11} b_{22} + a_{12} b_{21}, \\ a(x) = (x a_{11} + a_{12})/D_0, \\ b(x) = b_{21}(x-1) - b_{22}, \\ \frac{cb(x)}{B_1(b)} + \frac{da(x)}{B_2(a)} = \frac{(b_{21} + b_{22})c + a_{12}d}{D_0} + x \frac{a_{11}d - b_{21}c}{D_0} \end{cases}$$

(see also (2.10), (2.11), and (2.12) with $p(x) \equiv 1$).

If we put $k := \sqrt{-\lambda}k$ we get the boundary value problem

$$(2.23) \quad \begin{cases} Ly = y'' + k^2 y = f(t, y(t)), & 0 < t < 1, \\ B_1 y = a_{11} y(0) - a_{12} y'(0) = c, \\ B_2 y = b_{21} y(1) + b_{22} y'(1) = d. \end{cases}$$

The integral equation formulation of this problem is given by (2.20)

where

$$(2.24) \quad \begin{cases} D_k = \frac{a_{11} b_{21}}{k} \sin(k) + a_{11} b_{22} \cos(k) + a_{12} b_{21} \cos(k) - \\ \quad - k a_{12} b_{22} \sin(k), \\ a(x) = \left[\frac{a_{11}}{k} \sin(kx) + a_{12} \cos(kx) \right] / D_k, \\ b(x) = \frac{b_{21}}{k} \sin(k(x-1)) - b_{22} \cos(k(x-1)), \end{cases}$$

and we must have $D_k \neq 0$. The eigenvalues λ of

$$(2.25) \quad \begin{cases} Ly + \lambda y = 0 \\ B_1 y = B_2 y = 0 \end{cases}$$

are the zeroes of the transcendental equation $D_{\sqrt{\lambda}} = 0$.

We content ourselves now with considering a non-symmetric differential equation

$$(2.26) \quad \begin{cases} Ly = p_0 y'' + p_1 y' + p_2 y = f(t, y(t)), & 0 < t < 1, \\ B_1 y = a_{11} y(0) - a_{12} y'(0) = c, \\ B_2 y = b_{21} y(1) + b_{22} y'(1) = d, \end{cases}$$

where $a_{11}, a_{12}, b_{21}, b_{22} \geq 0$, $a_{11} + b_{21} > 0$, $a_{11} + a_{12} > 0$,

and $b_{21} + b_{22} > 0$, p_0, p_1, p_2, f are continuous and $p_0 > 0$ on the

interval $[0, 1]$. Let $\mathcal{B} = \{ y \in C^2[0, 1] \mid B_1 y = B_2 y = 0 \}$.

The differential equation in (2.26) can be transformed into a symmetric differential equation by

$$(2.27) \quad \frac{1}{p_0} \exp\left(\int_0^x \frac{p_1}{p_0} dt\right) Ly = \frac{1}{p_0} \exp\left(\int_0^x \frac{p_1}{p_0} dt\right) f(x, y(x)).$$

We get

$$(2.28) \quad (py')' = f_1(x, y(x)),$$

where

$$(2.29) \quad \begin{cases} p(x) = \exp\left(\int_0^x \frac{p_1(t)}{p_0(t)} dt\right) \\ q(x) = \frac{p_0(x)}{p_0(x)} \exp\left(\int_0^x \frac{p_1(t)}{p_0(t)} dt\right) \\ f_1(x, y(x)) = \frac{1}{p_0(x)} \exp\left(\int_0^x \frac{p_1(t)}{p_0(t)} dt\right) f(x, y(x)). \end{cases}$$

However we shall not consider this transformation here.

We assume that zero is not an eigenvalue of $(L, \mathcal{B})$, then Green's function exists and is unique (see Stakgold [23] p. 66) and it is given by (2.3). However in this case $(p_1 \neq p_0)$ C is not a constant. We have

$$(2.30) \quad C = p_0(t)W(a, b, t)$$

where

$$W(a, b, t) = \begin{vmatrix} b'(t) & a'(t) \\ b(t) & a(t) \end{vmatrix}$$

is the Wronskian of a, b . Put

$$(2.31) \quad \begin{cases} c(t) = a(t)/(p_0(t)W(a, b, t)) \\ d(t) = b(t)/(p_0(t)W(a, b, t)), \end{cases}$$

then Green's function for $(L, \mathcal{B})$ is

$$(2.32) \quad G(x,t) = \begin{cases} b(x)c(t), & x \geq t, \\ a(x)d(t), & x \leq t, \end{cases}$$

and the integral equation formulation of the boundary value problem (2.26) is

$$(2.33) \quad y(x) = \int_0^1 G(x,t)f(t,y(t))dt + \frac{cb(x)}{B_2(b)} + \frac{da(x)}{B_1(a)}, \quad 0 \leq x \leq 1,$$

where $G(x,t)$ is defined by (2.3), (2.31), and (2.32).

3. DETERMINANT FORMULATION OF CONTINUOUS BOUNDARY VALUE PROBLEMS

Consider the boundary value problem (2.1). Suppose that

$0 \leq x_1 < x_2 < x_3 \leq 1$. By (2.5) we have

$$(3.1) \quad y(x_i) - b(x_i) \int_0^{x_1} a(t)f(t,y(t))dt - a(x_i) \int_{x_3}^1 b(t)f(t,y(t))dt - \\ - \int_{x_1}^{x_3} G(x_i,t)f(t,y(t))dt - \frac{cb(x_i)}{B_1(b)} - \frac{da(x_i)}{B_2(a)} = 0,$$

for $i = 1, 2, 3$. Put

$$\begin{cases} X = \int_0^{x_1} a(t)f(t,y(t))dt + \frac{c}{B_1(b)}, \\ Y = \int_{x_3}^1 b(t)f(t,y(t))dt + \frac{d}{B_2(a)}, \\ G_i = y(x_i) - \int_{x_1}^{x_3} G(x_i,t)f(t,y(t))dt, \quad i = 1, 2, 3. \end{cases}$$

Then we get

$$\begin{cases} b(x_1)X + a(x_1)Y = G_1 \\ b(x_2)X + a(x_2)Y = G_2 \\ b(x_3)X + a(x_3)Y = G_3. \end{cases}$$

Hence we have an overdetermined linear system with three equations and two unknowns, X and Y . If the integral equation (2.5) has a solution y , then all three equations must be satisfied. This means that the three vectors

$$\begin{bmatrix} b(x_1) \\ b(x_2) \\ b(x_3) \end{bmatrix}, \begin{bmatrix} a(x_1) \\ a(x_2) \\ a(x_3) \end{bmatrix}, \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix},$$

are linearly dependent. Hence we must have

$$(3.2) \quad \begin{vmatrix} b(x_1) & a(x_1) & y(x_1) - \int_{x_1}^{x_3} G(x_1, t) f(t, y(t)) dt \\ b(x_2) & a(x_2) & y(x_2) - \int_{x_1}^{x_3} G(x_2, t) f(t, y(t)) dt \\ b(x_3) & a(x_3) & y(x_3) - \int_{x_1}^{x_3} G(x_3, t) f(t, y(t)) dt \end{vmatrix} = 0.$$

If $0 \leq x_1 \leq 1$, we obtain, by the same method, from (3.1)

$$(3.3) \quad \begin{vmatrix} a(0) & y(0) - \int_0^{x_1} G(0, t) f(t, y(t)) dt - cb(0)/B_1(b) \\ a(x_1) & y(x_1) - \int_0^{x_1} G(x_1, t) f(t, y(t)) dt - cb(x_1)/B_1(b) \end{vmatrix} = 0,$$

and if $0 \leq x_3 < 1$ we have

$$(3.4) \quad \begin{vmatrix} b(x_3) & y(x_3) - \int_{x_3}^1 G(x_3, t) f(t, y(t)) dt - da(x_3)/B_2(a) \\ b(1) & y(1) - \int_{x_3}^1 G(1, t) f(t, y(t)) dt - da(1)/B_2(a) \end{vmatrix} = 0.$$

Put

$$D(t_1, t_2) = \begin{vmatrix} b(t_1) & a(t_1) \\ b(t_2) & a(t_2) \end{vmatrix}, \quad 0 \leq t_1 < t_2 \leq 1.$$

Theorem 3.1

If $q \leq 0$ in equation (2.1) then $D(t_1, t_2) < 0$ for $0 \leq t_1 < t_2 \leq 1$. Otherwise $D(t_1, t_2) < 0$ if the difference $t_2 - t_1$ is sufficiently small.

Proof: If $q \leq 0$ put

$$D(h) = - \begin{vmatrix} b(x) & a(x) \\ b(x+h) & a(x+h) \end{vmatrix}, \quad x, x+h \in [0, 1].$$

We have $D(0) = 0$, $D'(0) = a(x)b'(x) - a'(x)b(x) = 1/p(x) > 0$, and also $L_h D(h) = 0$. Suppose that $D(h_0) = 0$, where $x + h_0 \leq 1$. Then by the maximum principle (see Protter and Weinberger [19]) $D(h) \equiv 0$, but $D'(0) > 0$. Hence $D(h) > 0$ for $h > 0$. In the other case we have

$$D(h)/h = a(x)b'(x) - a'(x)b(x) + O(h) = 1/p(x) + O(h) > 0$$

for h sufficiently small since $p > 0$.

Let G_n denote the grid $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$. Put $x_1 := x_{i-1}$, $x_2 := x_i$, and $x_3 := x_{i+1}$ in equation (3.2) and assume that Theorem 3.1 holds for $D(x_{i-1}, x_i)$, $i = 1, \dots, n$.

By determinant manipulation equation (3.2) can be written as

$$\begin{aligned}
 (3.5) \quad & \frac{-y(x_{i-1})}{D(x_{i-1}, x_i)} + \frac{D(x_{i-1}, x_{i+1})}{D(x_{i-1}, x_i)D(x_i, x_{i+1})} y(x_i) + \frac{-y(x_{i+1})}{D(x_i, x_{i+1})} = \\
 & = \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) f(t, y(t)) dt, \quad i = 1, \dots, n-1,
 \end{aligned}$$

where

$$(3.6) \quad \Delta_i(t) = \begin{cases} D(x_{i-1}, t)/D(x_{i-1}, x_i), & x_{i-1} \leq t \leq x_i, \\ D(t, x_{i+1})/D(x_i, x_{i+1}), & x_i \leq t \leq x_{i+1}. \end{cases}$$

If $a_{12} = 0$ then equation (3.3) can be written as $y(0) = c/a_{11}$ and if $b_{22} = 0$ then equation (3.4) can be written as $y(1) = d/b_{21}$. If $a_{12} > 0$ then $a(0) \neq 0$ and equation (3.3) can be written as

$$(3.7) \quad \frac{a(x_1)}{a(0)D(0, x_1)} y(0) + \frac{-y(x_1)}{D(0, x_1)} = \int_0^{x_1} \Delta_0(t) f(t, y(t)) dt + \frac{c}{a(0)B_1(b)},$$

where

$$(3.8) \quad \Delta_0(t) = D(t, x_1)/D(0, x_1), \quad 0 \leq t \leq x_1.$$

If $b_{22} > 0$ then $b(1) \neq 0$ and equation (3.4) can be written as

$$\begin{aligned}
 (3.9) \quad & \frac{-y(x_{n-1})}{D(x_{n-1}, 1)} + \frac{b(x_{n-1})}{b(1)D(x_{n-1}, 1)} y(1) = \int_{x_{n-1}}^1 \Delta_n(t) f(t, y(t)) dt + \\
 & + \frac{d}{b(1)B_2(a)},
 \end{aligned}$$

where

$$(3.10) \quad \Delta_n(t) = D(x_{n-1}, t)/D(x_{n-1}, 1), \quad x_{n-1} \leq t \leq 1.$$

We call (3.5), (3.7), and (3.9) the determinant formulation of the boundary value problem (2.1). If $a_{12} = 0$ we omit equation (3.7)

and substitute the boundary condition $y(0) = c/a_{11}$ into equation (3.7), and if $b_{22} = 0$ we omit equation (3.9) and substitute the boundary condition $y(1) = d/b_{21}$ into equation (3.7). We see that the left hand side of the system (3.5), (3.7), and (3.9) define a symmetric tridiagonal matrix. The nonsingularity of this matrix is very important for the equivalence of the determinant formulation and the integral equation on the given grid. When we talk about the matrix of the system (3.5), (3.7), and (3.9) we always mean the tridiagonal matrix of the left hand side.

Depending on the boundary conditions B_1 and B_2 we can extend the system (3.5) with (3.7) or (3.9). Hence we have 4 matrices to study. However if $B_1 y = a_{11} y(0)$ then the coefficients of the first row of (3.5) have the same structure as equation (3.7) since

$a(0) = 0$, $b(0) \neq 0$ give $D(0, x_2)/D(0, x_1) = a(x_2)/a(x_1)$. If

$B_2 y = b_{21} y(1)$ then, since $b(1) = 0$, $a(1) \neq 0$, we have

$D(x_{n-2}, 1)/D(x_{n-1}, 1) = b(x_{n-2})/b(x_{n-1})$, and the last row of (3.5) has the same structure as (3.9).

Consider (3.5) and put $x_i := x$, $x_{i-1} < x < x_{i+1}$. We get

$$\begin{aligned} y(x) &= \frac{D(x, x_{i+1})}{D(x_{i-1}, x_{i+1})} y(x_{i-1}) + \frac{D(x_{i-1}, x)}{D(x_{i-1}, x_{i+1})} y(x_{i+1}) + \\ &+ \int_{x_{i-1}}^x \frac{D(x, x_{i+1}) D(x_{i-1}, t)}{D(x_{i-1}, x_{i+1})} f(t, y(t)) dt + \\ &+ \int_x^{x_{i+1}} \frac{D(x_{i-1}, x) D(t, x_{i+1})}{D(x_{i-1}, x_{i+1})} f(t, y(t)) dt, \quad x_{i-1} \leq x \leq x_{i+1}. \end{aligned}$$

Put

$$\begin{cases} a(x) = D(x_{i-1}, x)/D(x_{i-1}, x_{i+1}), \\ b(x) = D(x, x_{i+1}), \end{cases}$$

then the determinant formulation of the integral equation above at

the points $x_{i-1} \leq \xi_1 < \xi_2 < \xi_3 \leq x_{i+1}$ is given by (3.5). With

$$T(x, t) = \begin{vmatrix} D(x, x_{i+1}) & D(x_{i-1}, x)/D(x_{i-1}, x_{i+1}) \\ D(t, x_{i+1}) & D(x_{i-1}, t)/D(x_{i-1}, x_{i+1}) \end{vmatrix}$$

we can write

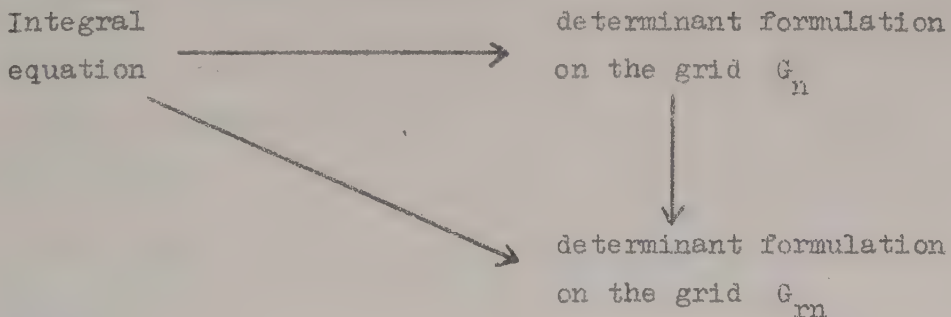
$$\begin{aligned} & \frac{-y(\xi_1)}{T(\xi_1, \xi_2)} + \frac{T(\xi_1, \xi_3)}{T(\xi_1, \xi_2)T(\xi_2, \xi_3)} y(\xi_2) + \frac{-y(\xi_3)}{T(\xi_2, \xi_3)} = \\ & = \int_{\xi_1}^{\xi_2} \frac{T(\xi_1, t)}{T(\xi_1, \xi_2)} f(t, y(t)) dt + \int_{\xi_2}^{\xi_3} \frac{T(t, \xi_3)}{T(\xi_2, \xi_3)} f(t, y(t)) dt. \end{aligned}$$

But it is easy to see that $T(x, t) = D(x, t)$ so the above equation is the determinant formulation of the integral equation (2.5) at

the points ξ_1, ξ_2, ξ_3 . Let $G_n = \{x_i \mid 0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1\}$ be a subset of

$$G_{rn} = \bigcup_{i=0}^{n-1} \{x_{i,j} \mid x_i = x_{i,0} < x_{i,1} < \dots < x_{i,r-1} < x_{i,r} = x_{i+1}\}.$$

Then we have the following commutative diagram



This is an important property which is used in the construction of Newton-like method for the numerical solution of boundary value problems (see Aulin [2], [5]).

Theorem 3.2

If p and q are constant in the interval $[0,1]$, then for $0 \leq x < x+h \leq 1$ the determinant $D(x, x+h)$ only depends on h , i.e. $D(x, x+h) = D(h)$.

Proof: Assume that $p(x) \equiv 1$, then

$$D(0) = 0, D'_h(0) = -(a(x)b'(x) - a'(x)b(x)) = -1,$$

and $L_h D(h) = 0$. Hence we have an initial value problem and it is clear that the unique solution is independent of x .

For p, q constant, and $x_i = i/n$, $i = 0, \dots, n$, equation (3.5) can be written in the more simple form

$$(3.11) \quad -y\left(\frac{i-1}{n}\right) + \frac{D(2h)}{D(h)} y\left(\frac{i}{n}\right) - y\left(\frac{i+1}{n}\right) = h \int_{-1}^1 D(h(1-|t|)) f\left(\frac{i+t}{n}, y\left(\frac{i+t}{n}\right)\right) dt,$$

for $i = 1, \dots, n-1$, $h = 1/n$.

Example 3.1

Consider the boundary value problem

$$(3.12) \quad \begin{cases} y'' = f(t, y(t)), & 0 < t < 1, \\ y(0) = c, & y(1) = d. \end{cases}$$

By (2.22) we have

$$\begin{cases} a(x) = x, \\ b(x) = x - 1. \end{cases}$$

With $x_i = i/n$, $i = 0, \dots, n$, and $h = 1/n$, we have

$$D(h) = \begin{vmatrix} \frac{1}{n} - 1 & \frac{1}{n} \\ \frac{1}{n} + h - 1 & \frac{1}{n} + h \end{vmatrix} = -h.$$

Hence the determinant formulation of the boundary value problem (3.12) is

$$(3.13) \quad y\left(\frac{i-1}{n}\right) - 2y\left(\frac{i}{n}\right) + y\left(\frac{i+1}{n}\right) = h^2 \int_{-1}^1 (1-|t|) f\left(\frac{i+t}{n}, y\left(\frac{i+t}{n}\right)\right) dt,$$

for $i = 1, \dots, n-1$.

The following example is of interest in alternating direction implicit methods for the numerical solution of elliptic boundary value problems.

Example 3.2

Consider the boundary value problem

$$(3.14) \quad \begin{cases} y'' - k^2 y = f(t, y(t)), & 0 < t < 1, \quad k > 0, \\ y(0) = c, \quad y(1) = d. \end{cases}$$

By (2.21) we have

$$\begin{cases} a(x) = \frac{\sinh(kx)}{k \sinh(k)} \\ b(x) = \sinh(k(x-1)). \end{cases}$$

With the same grid as in example 3.1 we get

$$D(h) = -\sinh(kh)/k.$$

Hence the determinant formulation of (3.14) is

$$(3.15) \quad y\left(\frac{i-1}{n}\right) - 2\cosh\left(\frac{k}{n}\right)y\left(\frac{i}{n}\right) + y\left(\frac{i+1}{n}\right) = \\ = h \int_{-1}^1 \frac{1}{k} \sinh(kh(1-|t|)) f\left(\frac{i+t}{n}, y\left(\frac{i+t}{n}\right)\right) dt,$$

for $i = 1, \dots, n-1$.

If we put $k := \sqrt{-1}k$ in (3.14) we get the boundary value problem

$$(3.16) \quad \begin{cases} y'' + k^2 y = f(t, y(t)), & 0 < t < 1, \quad k > 0, \\ y(0) = c, \quad y(1) = d, \end{cases}$$

and the determinant formulation of (3.16) is

$$\begin{aligned}
 y\left(\frac{i-1}{n}\right) - 2\cos\left(\frac{k}{n}\right)y\left(\frac{i}{n}\right) + y\left(\frac{i+1}{n}\right) = \\
 = h \int_{-1}^1 \frac{1}{k} \sin(kh(1-|t|)) f\left(\frac{i+t}{n}, y\left(\frac{i+t}{n}\right)\right) dt,
 \end{aligned}$$

for $i = 1, \dots, n-1$.

If we let $k \rightarrow 0$ in (3.15) we get (3.13).

We will now give a simple criterion for the nonsingularity of the tridiagonal matrix defined by (3.5).

Theorem 3.3

Given two sequences $\{a_k\}_{k=1}^{n-1}$, $\{b_k\}_{k=1}^{n-1}$ completed with $a_0 = b_n = 0$, $b_0 \neq 0$, and $a_n \neq 0$. Put

$$d(i, j) = \begin{vmatrix} b_i & a_i \\ b_j & a_j \end{vmatrix}, \quad 0 \leq i < j \leq n.$$

Let $G = (g_{ij})$ be the $(n-1) \times (n-1)$ matrix defined by

$$g_{ij} = \begin{cases} b_i a_j, & i \geq j, \\ a_i b_j, & i \leq j, \end{cases}$$

for $i, j = 1, \dots, n-1$. Then G is nonsingular if and only if $d(k, k+1) \neq 0$ for $k = 0, 1, \dots, n-1$.

Proof: If G is nonsingular consider its determinant

$$\det G = b_{n-1} \begin{vmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 & \dots & a_1 b_{n-1} \\ b_2 a_1 & a_2 b_2 & a_2 b_3 & \dots & a_2 b_{n-1} \\ b_3 a_1 & b_3 a_2 & a_3 b_3 & \dots & a_3 b_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-2} a_1 & b_{n-2} a_2 & b_{n-2} a_3 & \dots & a_{n-2} b_{n-1} \\ a_1 & a_2 & a_3 & \dots & a_{n-1} \end{vmatrix}.$$

Multiply the last row by b_i and subtract it from the i :th row

for $i = 1, \dots, n-2$, we get

$$\det G = b_{n-1} \begin{vmatrix} 0 & -d(1,2) & -d(1,3) & \dots & -d(1,n-1) \\ 0 & 0 & -d(2,3) & \dots & -d(2,n-1) \\ 0 & 0 & 0 & \dots & -d(3,n-1) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -d(n-2,n-1) \\ a_1 & a_2 & a_3 & \dots & a_{n-1} \end{vmatrix} =$$

$$= a_1 b_{n-1} \prod_{i=1}^{n-2} d(k, k+1) \neq 0.$$

But $b_0 a_1 = d(0,1)$ and $a_n b_{n-1} = d(n-1,n)$. Hence $d(k, k+1) \neq 0$ for $k = 0, 1, \dots, n-1$. On the other hand if all $d(k, k+1) \neq 0$ put

$$T = \begin{bmatrix} \frac{d(0,2)}{d(0,1)d(1,2)} & -\frac{1}{d(1,2)} & & & 0 \\ -\frac{1}{d(1,2)} & \frac{d(1,3)}{d(1,2)d(2,3)} & -\frac{1}{d(2,3)} & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -\frac{1}{d(n-2,n-1)} & \frac{d(n-2,n)}{d(n-2,n-1)d(n-1,n)} & \end{bmatrix}.$$

Then $TG = I$ and $G^{-1} = T$.

Corollary

If $q \leq 0$ or $x_{i+1} - x_i$, $i = 0, \dots, n-1$, are sufficiently small then the tridiagonal matrix T defined by equation (3.5) is nonsingular and $T^{-1} = G = (g_{ij})$, where

$$(3.17) \quad g_{ij} = \begin{cases} b(x_i)a(x_j), & i \geq j, \\ a(x_i)b(x_j), & i \leq j, \end{cases}$$

for $i, j = 1, \dots, n-1$, $a(0) = b(1) = 0$.

Proof: We have $a(0) = b(1) = 0$ and since $B_1(b) \neq 0$, $B_2(a) \neq 0$ we

have $b(0) \neq 0$, $a(1) \neq 0$. By Theorem 3.1 $D(x_1, x_{1+1}) < 0$ if $q \leq 0$ or $x_{1+1} - x_1$ is sufficiently small. Put $d(1, 1+1) = D(x_1, x_{1+1})$ and apply the theorem.

Remark.

If $a_{12} > 0$ we include equation (3.7) in the system (3.5). Then complete the sequences $\{a(x_i)\}_{i=0}^n$, $\{b(x_i)\}_{i=0}^n$ with $a(x_{-1}) = 0$ and $b(x_{-1}) = 1$. Since $a(0) \neq 0$ we get $D(x_{-1}, 0) = a(0) \neq 0$.

If $b_{22} > 0$ we include equation (3.9) in the system (3.5). Then complete the sequences $\{a(x_i)\}_{i=0}^n$, $\{b(x_i)\}_{i=0}^n$ with $b(x_{n+1}) = 0$, $a(x_{n+1}) = 1$. Since $b(1) \neq 0$ we get $D(1, x_{n+1}) = b(1) \neq 0$. Hence the system (3.5) extended with (3.7) or (3.9) also has a non-singular matrix T if the conditions of the Corollary hold.

Moreover we have $T^{-1} = G = (g_{ij})$, where $i, j \geq 0$ if $a_{12} > 0$, $i, j \leq n$ if $b_{22} > 0$, and g_{ij} is defined by (3.17).

To improve Theorem 3.1 we will now consider the eigenvalue problem

$$(3.18) \quad \begin{cases} (py')' + (q + \lambda)y = 0, & 0 < t < 1 \\ y(0) = y(1) = 0. \end{cases}$$

We need a comparison theorem.

Lemma 3.4 (Sturm comparison theorem)

Let $p(x) > 0$ and $q_1(x) > q(x)$ in the differential equations

$$(py')' + qy = 0, \quad (pz')' + q_1z = 0.$$

Then, between any two zeros of a nontrivial solution $y(x)$ of the first equation there lies at least one zero of every solution of the second equation.

Proof: (See Birkhoff and Rota [7] theorem 3 p. 290.)

Lemma 3.5

The eigenvalue problem (3.18) has an infinite sequence of real eigenvalues $\lambda_0 < \lambda_1 < \dots$ with $\lim_{n \rightarrow \infty} \lambda_n = \infty$. The eigenfunction belonging to the eigenvalue λ_n has exactly n zeros in the interval $0 < x < 1$.

Proof: (See Birkhoff and Rota [7] theorem 5 p. 296.)

Theorem 3.6

Assume that the first eigenvalue λ_0 of (3.18) is greater than zero. Then $D(x_1, x_2) \neq 0$ for $0 \leq x_1 < x_2 \leq 1$.

Proof: Consider the determinant

$$D(x_1, t), \quad 0 \leq t \leq 1,$$

as a function of t . We have $L_t D(x_1, t) = 0$, $0 < t < 1$, and $D(x_1, x_1) = 0$. Assume that $D(x_1, x_2) = 0$ for $x_1 < x_2 \leq 1$. Let y_0 be the eigenfunction belonging to the first eigenvalue $\lambda_0 > 0$. Since $\lambda_0 + q > q$ and $D(x_1, x_1) = D(x_1, x_2) = 0$ we must have $y_0(x_0) = 0$ for some x_0 between x_1 and x_2 by Lemma 3.4. But by Lemma 3.5 y_0 does not have a zero in the interval $(0, 1)$. Hence we must have $D(x_1, x_2) \neq 0$.

Theorem 3.7

If the first eigenvalue of (3.18) is greater than zero then the integral equation and the determinant formulation of the boundary value problem (2.1) are equivalent on the grid G_n , i.e. the tridiagonal matrix of the determinant formulation is nonsingular and $T^{-1} = G = (g_{ij})$, where

$$(3.19) \quad g_{ij} = \begin{cases} b(x_i)a(x_j), & i \geq j, \\ a(x_i)b(x_j), & i \leq j, \end{cases}$$

and

$$i, j \geq \begin{cases} 0 & \text{if } a_{12} > 0, \\ 1 & \text{if } a_{12} = 0, \end{cases} \quad i, j \leq \begin{cases} n & \text{if } b_{22} > 0, \\ n-1 & \text{if } b_{22} = 0. \end{cases}$$

Proof: The conclusions of the theorem follow from Theorems 3.6, 3.3, and the Remark of the Corollary of Theorem 3.3.

Example 3.3

Consider the boundary value problem

$$\begin{cases} Ly = y'' + k^2 y = f, & k \neq n, \quad n = 1, 2, \dots \\ y(0) = y(1) = 0. \end{cases}$$

By (2.24) we have

$$\begin{cases} a(x) = \sin(kx)/(k \sin(k)) \\ b(x) = \sin(k(x-1)). \end{cases}$$

Hence

$$D(x_i, x_{i+1}) = \sin(k(x_i - x_{i+1})).$$

The Wronskian of a and b is

$$W(a, b, x) = ab' - a'b = 1/p(x) = 1.$$

If $0 < k < \pi$ we have $D(x_i, x_{i+1}) \neq 0$ for $0 \leq x_i < x_{i+1} \leq 1$.

Let $k = 3\pi/2$. Then

$$D(x_i, x_{i+1}) = 0 \quad \text{if} \quad x_{i+1} - x_i = 2/3.$$

Thus on the grid $0, \frac{1}{6}, \frac{5}{6}, 1$ the determinant formulation of the boundary value problem does not exist in the form given by (3.5). We have

$$D(\frac{1}{6}, \frac{5}{6}) = -\sin(\frac{3\pi}{2} \frac{4}{6}) = -\sin(\pi) = 0.$$

The first eigenvalues of the eigenvalue problem

$$\begin{cases} y'' + (\lambda + 9\pi^2/4)y = 0 \\ y(0) = y(1) = 0 \end{cases}$$

are $\lambda_0 = -5\pi^2/4$, $\lambda_1 = 7\pi^2/4$.

In order to give a general treatment of the determinant formulation we shall now introduce distributions. We give a brief outline of this theory. Excellent expositions are found in Hörmander [14], [15], Schwartz [21], [22], or Treves [25] (and in many other texts).

If Ω is an open set in $\mathbb{R}^n$ we denote by $C^k(\Omega)$ the set of k times continuously differentiable functions in Ω , where k may be any integer ≥ 0 or $+\infty$.

If u is a continuous function in Ω the support of u , denoted $\text{supp } u$, is defined to be the closure of the set $\{x \mid x \in \Omega, u(x) \neq 0\}$. By $C_0^k(\Omega)$ we shall denote the set of elements in $C^k(\Omega)$ having compact support.

A distribution u in Ω is a linear form on $C_0^\infty(\Omega)$ such that for every compact set $K \subset \Omega$ there exist constants C and k such that

$$(3.20) \quad |u(\varphi)| \leq C \sum_{j \leq k} \sup \left| \frac{d^j \varphi}{dx^j} \right|, \quad \varphi \in C_0^k(K).$$

The set of all distributions in Ω is denoted by $\mathcal{D}'(\Omega)$. If the same integer k can be used in (3.20) for every K we say that u is of order $\leq k$ and we denote the set of such distributions by $\mathcal{D}'^k(\Omega)$.

A linear form u on $C_0^\infty(\Omega)$ is a distribution if and only if $u(\varphi_j) \rightarrow 0$ when $j \rightarrow \infty$ if φ_j is any sequence in $C_0^\infty(\Omega)$ converging to 0 in the sense that

$$(i) \quad \sup |\varphi_j^{(k)}| \rightarrow 0 \text{ for every fixed } k$$

and

$$(ii) \quad \text{supp } \varphi_j \subset K \text{ for all } j \text{ and some fixed compact set } K \subset \Omega.$$

We identify the space of continuous functions in Ω with a subspace of $\mathcal{D}'(\Omega)$ by assigning to each continuous f the distribution

$$(3.21) \quad C_0^\infty(\Omega) \ni \varphi \rightarrow \int_{\Omega} f \varphi dx$$

which we also denote by f . More generally we can make this identification when $f \in L_{loc}^1(\Omega)$, the space of functions which are integrable on compact subsets of Ω , modulo those which vanish almost everywhere. We can also identify arbitrary measures with distributions of order 0, for if $u \in \mathcal{D}'^k(\Omega)$ we can in a unique way extend u to a linear form on $C_0^k(\Omega)$ such that (3.20) remains valid for all $\varphi \in C_0^k(K)$ and some constant C (see Hörmander [15]). Since a measure can be defined to be a linear form on $C_0^0(\Omega)$ with the continuity property (3.20) for $k = 0$ we have now identified $\mathcal{D}'^0(\Omega)$ with the space of measures in Ω . The identification of functions with distributions made above agrees with the usual identification of f with the measure $f dx$ used in integration theory.

A linear form u on $C_0^m(\Omega)$ is a distribution in $\mathcal{D}'^m(\Omega)$ if and only if $u(\varphi_j) \rightarrow 0$ when $j \rightarrow \infty$ if φ_j is any sequence in $C_0^m(\Omega)$ converging to 0 in the sense that

$$(i) \quad \sup |\varphi_j^{(k)}| \rightarrow 0 \text{ for every } k \leq m$$

and

$$(ii) \quad \text{supp } \varphi_j \subset K \text{ for all } j \text{ and some fixed compact set } K \subset \Omega.$$

If a_j are complex numbers and $u_j \in \mathcal{D}'(\Omega)$ we set

$$(a_1 u_1 + a_2 u_2)(\varphi) = a_1 u_1(\varphi) + a_2 u_2(\varphi).$$

Thus $\mathcal{D}'(\Omega)$ is a vector space and so is $\mathcal{D}'^k(\Omega)$. If $\hat{\Omega} \subset \Omega$

we can restrict every distribution $u \in \mathcal{D}'(\Omega)$ to a distribution $\hat{u}$ in $\hat{\Omega}$,

$$\hat{u}(\varphi) = u(\varphi), \quad \varphi \in C_0^\infty(\hat{\Omega}).$$

If $u \in \mathcal{D}'(\Omega)$ the support of u , denoted $\text{supp } u$, is the set of points in Ω having no open neighborhood to which the restriction of u is 0.

If $u \in \mathcal{D}'(\Omega)$ the derivative $\frac{du}{dx}$ is defined by the equation

$$(3.22) \quad \frac{du}{dx}(\varphi) = -u\left(\frac{d\varphi}{dx}\right), \quad \varphi \in C_0^\infty(\Omega),$$

and if $f \in C^\infty(\Omega)$ we define

$$(3.23) \quad (fu)(\varphi) = u(f\varphi), \quad \varphi \in C_0^\infty(\Omega).$$

It is clear that (3.22) and (3.23) define new distributions.

Example 3.4

The function

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases}$$

is called the Heaviside function. Now consider H as a distribution.

The definition of $\frac{dH}{dx}$ means that $H'(\varphi) = -H(\varphi')$, and

$$-H(\varphi') = -\int_0^\infty \varphi'(x) dx = [-\varphi]_0^\infty = \varphi(0).$$

Hence $H' = \delta$, the Dirac measure at 0, defined by $\delta(\varphi) = \varphi(0)$.

If we consider H as a function then

$$\int_0^\infty H'(x)\varphi(x) dx = 0$$

since $H'(x) = 0$ in $(0, \infty)$.

If $u \in \mathcal{D}'(\Omega)$ and $\text{supp } u$ is compact then there is a unique linear form $\tilde{u}$ on $C^\infty(\Omega)$ such that $\tilde{u}(\varphi) = u(\varphi)$ if $\varphi \in C_0^\infty(\Omega)$ and $\tilde{u}(\varphi) = 0$ if $\text{supp } u \cap \text{supp } \varphi = \emptyset$. The estimate

$$(3.24) \quad |\tilde{u}(\varphi)| \leq c \sum_{j \leq k} \sup_K |\varphi^{(j)}|, \quad \varphi \in C^\infty(\Omega)$$

is valid for some constants C, k if K is any compact neighborhood of $\text{supp } u$ in Ω . Conversely let $\tilde{u}$ be a linear form on $C^\infty(\Omega)$ satisfying (3.24) where K is a compact subset of Ω . Then the restriction of $\tilde{u}$ to $C_0^\infty(\Omega)$ is a distribution $u \in \mathcal{D}'(\Omega)$ with $\text{supp } u \subset K$, and $u = 0$ implies $\tilde{u} = 0$. (See Hörmander [15] theorems 2.3.1 and 2.3.2.) The preceding results allow us to identify the space of distributions in Ω with compact support and the space of all linear form on $C^\infty(\Omega)$ satisfying (3.24) for some compact set $K \subset \Omega$. We denote this space by $\mathcal{E}'(\Omega)$.

A linear form u on $C^\infty(\Omega)$ is a distribution with compact support if and only if $u(\varphi_j) \rightarrow 0$ for every sequence $\varphi_j \in C^\infty(\Omega)$ tending to 0 in the sense that $\varphi_j^{(k)} \rightarrow 0$ uniformly on any compact subset of Ω for every fixed k .

If the same integer k can be used in (3.24) for every K we say that u is of order $\leq k$ and we denote the set of such distributions by $\mathcal{E}^k(\Omega)$. Thus $\mathcal{E}^k(\Omega) = \mathcal{E}'(\Omega) \cap \mathcal{D}'^k(\Omega)$.

From now on we simplify the notations and write $u(\varphi)$ or $\langle u, \varphi \rangle$ instead of $\tilde{u}(\varphi)$ when $u \in \mathcal{E}'(\Omega)$ and $\varphi \in C^\infty(\Omega)$, or $u \in \mathcal{E}^k(\Omega)$ and $\varphi \in C^k(\Omega)$. For $u \in \mathcal{E}'(\Omega)$ and $f, \varphi \in C^\infty(\Omega)$ we still have

$$\langle fu, \varphi \rangle = \langle u, f\varphi \rangle, \quad \left\langle \frac{du}{dx}, \varphi \right\rangle = - \left\langle u, \frac{d\varphi}{dx} \right\rangle.$$

A linear form u on $C^m(\Omega)$ is a distribution with compact support if and only if $u(\varphi_j) \rightarrow 0$ for every sequence $\varphi_j \in C^m(\Omega)$ tending to 0 in the sense that $\varphi_j^{(k)} \rightarrow 0$ uniformly on any compact subset of Ω for every $k \leq m$.

Let Ω be an open neighborhood of $[0, 1]$, and

$$L = \sum_{j=0}^m a_j(x) (d/dx)^j,$$

where $a_j \in C^j(\Omega)$, $1/a_m \in C^m(\Omega)$, a differential operator in Ω .

It is clear that $LC^m(\Omega) \subset C^0(\Omega)$. If $f \in C^0(\Omega)$ then there exist

solutions u of $Lu = f$ and $u \in C^m(\Omega)$. Hence the equation is fulfilled in the classical sense. The mapping

$$L: C^m(\Omega) \rightarrow C^0(\Omega)$$

is surjective. If $f, g \in C_0^m(\Omega)$ we obtain by integration by parts

$$\int_{\Omega} Lf(x) g(x) dx = \int_{\Omega} f(x) L'g(x) dx$$

where

$$L' = \sum_{j=0}^m (-d/dx)^j a_j(x)$$

is called the transpose of L .

Consider the differential operator L , of the boundary value problem (2.1),

$$L = \frac{d}{dx} \left(p \frac{d}{dx} \right) + q.$$

We can continue p and $1/p$ to C^1 -functions on Ω and also q to a continuous function on Ω .

Let f be a continuous function with compact support in Ω . Then $f \in \mathcal{E}'^0(\Omega)$ and also $\mathcal{E}'^2(\Omega)$. We define Lf by ($L' = L$)

$$\langle Lf, \varphi \rangle = \langle f, L\varphi \rangle = \int_{\Omega} f L\varphi dx = \int_{\text{supp } f} f L\varphi dx,$$

for $\varphi \in C^2(\Omega)$. This makes sense since $L\varphi \in C^0(\Omega)$. Lf is a linear form on $C^2(\Omega)$. It is also continuous since if $\varphi_j^{(k)} \rightarrow 0$ uniformly on any compact subset of Ω for $k \leq 2$, we have

$$\langle Lf, \varphi_j \rangle = \int_{\Omega} f L\varphi_j dx = \int_{\text{supp } f} f L\varphi_j dx \rightarrow 0$$

when $j \rightarrow \infty$. Hence $Lf \in \mathcal{E}'^2(\Omega)$.

Consider the determinant formulation (3.5) of the boundary value problem (2.1). We assume that the first eigenvalue λ_0 of the eigenvalue problem

$$(3.25) \quad \begin{cases} Ly + \lambda y = 0 \\ y(0) = y(1) = 0 \end{cases}$$

is greater than zero. Then $D(x_i, x_{i+1}) \neq 0$ for $i = 0, 1, \dots, n-1$.

Put

$$\begin{cases} c_i = -1/D(x_{i-1}, x_i), & i = 1, \dots, n, \\ d_i = \frac{D(x_{i-1}, x_{i+1})}{D(x_{i-1}, x_i)D(x_i, x_{i+1})}, & i = 1, \dots, n-1. \end{cases}$$

Since $f(t, y(t)) = Ly$ equation (3.5) can be written as

$$(3.26) \quad c_i y(x_{i-1}) + d_i y(x_i) + c_{i+1} y(x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) Ly dt,$$

where $\Delta_i(t)$ is a continuous function, it is a C^2 -function in

(x_{i-1}, x_i) , (x_i, x_{i+1}) , and $\text{supp } \Delta_i = [x_{i-1}, x_{i+1}]$. Let δ_{x_i} be the

Dirac measure at $x_i \in \Omega$,

$$\langle \delta_{x_i}, \varphi \rangle = \varphi(x_i), \quad \varphi \in C^0(\Omega).$$

Then equation (3.26) can be written as

$$(3.27) \quad \langle c_i \delta_{x_{i-1}} + d_i \delta_{x_i} + c_{i+1} \delta_{x_{i+1}}, y \rangle = \langle \Delta_i, Ly \rangle,$$

where y is a solution of the boundary value problem (2.1).

Let $v(x) = \Delta_i'(x)$ for $x \neq x_{i-1}, x_i, x_{i+1}$. If $x_{i-1} < z < x_i$ then

$$\Delta_i'(x) = \Delta_i'(z) - \int_x^z v(t) dt, \quad x_{i-1} < x < z < x_i$$

which shows that $\Delta_i'(x_{i-1}+0)$ exists. Similarly $\Delta_i'(x_i-0)$, $\Delta_i'(x_i+0)$,

and $\Delta_i'(x_{i+1}-0)$ exist.

Now consider $\langle \Delta_i, Ly \rangle$ for $y \in C^2(\Omega)$. Integration by parts give $(\varepsilon > 0)$

$$\begin{aligned}
& \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) [(p(t)y'(t))' + q(t)y(t)] dt = \\
& = \lim_{\varepsilon \rightarrow 0} \int_{x_{i-1}+\varepsilon}^{x_i-\varepsilon} + \int_{x_i+\varepsilon}^{x_{i+1}-\varepsilon} \Delta_i(t) [(p(t)y'(t))' + q(t)y(t)] dt = \\
& = \lim_{\varepsilon \rightarrow 0} \left\{ \Delta_i(x_i-\varepsilon)p(x_i-\varepsilon)y'(x_i-\varepsilon) - \Delta_i(x_{i-1}+\varepsilon)p(x_{i-1}+\varepsilon)y'(x_{i-1}+\varepsilon) + \right. \\
& + \Delta_i(x_{i+1}-\varepsilon)p(x_{i+1}-\varepsilon)y'(x_{i+1}-\varepsilon) - \Delta_i(x_i+\varepsilon)p(x_i+\varepsilon)y'(x_i+\varepsilon) - \\
& - [p(x_i-\varepsilon)\Delta_i'(x_i-\varepsilon)y(x_i-\varepsilon) - p(x_{i-1}+\varepsilon)\Delta_i'(x_{i-1}+\varepsilon)y(x_{i-1}+\varepsilon) + \\
& + p(x_{i+1}-\varepsilon)\Delta_i'(x_{i+1}-\varepsilon)y(x_{i+1}-\varepsilon) - p(x_i+\varepsilon)\Delta_i'(x_i+\varepsilon)y(x_i+\varepsilon)] + \\
& + \int_{x_{i-1}+\varepsilon}^{x_i-\varepsilon} (L\Delta_i(t))y(t)dt + \int_{x_i+\varepsilon}^{x_{i+1}-\varepsilon} (L\Delta_i(t))y(t)dt \Big\} = \\
& = p(x_{i-1})\Delta_i'(x_{i-1}+0)y(x_{i-1}) - p(x_i)[\Delta_i'(x_i-0) - \Delta_i'(x_i+0)]y(x_i) - \\
& - p(x_{i+1})\Delta_i'(x_{i+1}-0)y(x_{i+1}), \quad y \in C^2(\Omega),
\end{aligned}$$

where

$$0 = \int_{x_{i-1}+\varepsilon}^{x_i-\varepsilon} (L\Delta_i(t))y(t)dt + \int_{x_i+\varepsilon}^{x_{i+1}-\varepsilon} (L\Delta_i(t))y(t)dt \rightarrow 0$$

when $\varepsilon \rightarrow 0$, since $L\Delta_i(t) = 0$ in (x_{i-1}, x_i) and (x_i, x_{i+1}) .

(In the sequel we will omit the limiting process and just carry out the integration by parts.) Hence we have

$$\begin{aligned}
(3.28) \quad L\Delta_i &= \Delta_i'(x_{i-1}+0)p(x_{i-1})\delta_{x_{i-1}} - \\
&- p(x_i)[\Delta_i'(x_i-0) - \Delta_i'(x_i+0)]\delta_{x_i} - p(x_{i+1})\Delta_i'(x_{i+1}-0)\delta_{x_{i+1}}
\end{aligned}$$

in $\mathcal{E}'(\Omega)$. Since the coefficients of the Dirac measures of (3.27)

are independent of y we must have

$$(3.29) \quad \begin{cases} c_i = p(x_{i-1}) \Delta_i'(x_{i-1}+0), \\ d_i = p(x_i) [\Delta_i'(x_i+0) - \Delta_i'(x_i-0)], \\ c_{i+1} = -p(x_{i+1}) \Delta_i'(x_{i+1}-0). \end{cases}$$

Now let y_1 and y_2 be two linearly independent solutions of $Ly = 0$ in $\mathcal{O}$. Put

$$(3.30) \quad \Delta_i(t) = \begin{vmatrix} y_2(x_{i-1}) & y_1(x_{i-1}) \\ y_2(t) & y_1(t) \end{vmatrix} / \begin{vmatrix} y_2(x_{i-1}) & y_1(x_{i-1}) \\ y_2(x_i) & y_1(x_i) \end{vmatrix},$$

for $x_{i-1} \leq t \leq x_i$, where

$$\begin{vmatrix} y_2(x_{i-1}) & y_1(x_{i-1}) \\ y_2(x_i) & y_1(x_i) \end{vmatrix} \neq 0$$

by (3.25). Then we have

$$(3.31) \quad \begin{cases} L\Delta_i(t) = 0, & x_{i-1} < t < x_i, \\ \Delta_i(x_{i-1}) = 0, & \Delta_i(x_i) = 1. \end{cases}$$

Further more $\Delta_i(t)$, $x_{i-1} \leq t \leq x_i$, is uniquely defined by (3.31) since if $\tilde{\Delta}_i$ is another solution of (3.31) then $u = \Delta_i - \tilde{\Delta}_i$ satisfies

$$(3.32) \quad \begin{cases} Lu = 0, & x_{i-1} < t < x_i, \\ u(x_{i-1}) = 0, & u(x_i) = 0. \end{cases}$$

By (3.25) this system only has the trivial solution $u(t) \equiv 0$.

By exactly the same method we can show that $\Delta_i(t)$ is uniquely defined in the interval $[x_i, x_{i+1}]$ by

$$(3.33) \quad \Delta_i(t) = \begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x_{i+1}) & y_1(x_{i+1}) \end{vmatrix} / \begin{vmatrix} y_2(x_i) & y_1(x_i) \\ y_2(x_{i+1}) & y_1(x_{i+1}) \end{vmatrix},$$

for $x_i \leq t \leq x_{i+1}$. Hence we have almost shown the following theorem.

Theorem 3.8

The continuous function $\Delta_i(t)$, $x_{i-1} \leq t \leq x_{i+1}$, and the constants c_i , c_{i+1} , d_i are uniquely defined by

$$(3.34) \quad \begin{cases} L\Delta_i(t) = 0, & x_{i-1} < t < x_i, \quad \Delta_i(x_{i-1}) = 0, \quad \Delta_i(x_i) = 1, \\ L\Delta_i(t) = 0, & x_i < t < x_{i+1}, \quad \Delta_i(x_i) = 1, \quad \Delta_i(x_{i+1}) = 0, \end{cases}$$

and

$$(3.35) \quad \begin{cases} c_i = p(x_{i-1})\Delta'_i(x_{i-1}+0), \\ d_i = p(x_i)[\Delta'_i(x_i+0) - \Delta'_i(x_i-0)], \\ c_{i+1} = -p(x_{i+1})\Delta'_i(x_{i+1}-0), \end{cases}$$

for $i = 1, \dots, n-1$, if the first eigenvalue of (3.25) is greater than zero.

Proof: It remains to show that

$$(3.36) \quad -p(x_{i+1})\Delta'_i(x_{i+1}-0) = p(x_i)\Delta'_i(x_i+0).$$

Let

$$\tilde{D}(x_i, x_{i+1}) = \begin{vmatrix} y_2(x_i) & y_1(x_i) \\ y_2(x_{i+1}) & y_1(x_{i+1}) \end{vmatrix}, \quad i = 0, 1, \dots, n-1.$$

Then

$$\begin{aligned} \Delta'_i(x_{i+1}-0) &= \begin{vmatrix} y'_2(x_{i+1}) & y'_1(x_{i+1}) \\ y_2(x_{i+1}) & y_1(x_{i+1}) \end{vmatrix} / \tilde{D}(x_i, x_{i+1}) = \\ &= W(y_1, y_2, x_{i+1}) / \tilde{D}(x_i, x_{i+1}), \end{aligned}$$

where $W(y_1, y_2, x_{i+1})$ is the Wronskian of y_1 and y_2 at the point x_{i+1} . We also have

$$\begin{aligned} \Delta'_{i+1}(x_i+0) &= \begin{vmatrix} y_2(x_i) & y_1(x_i) \\ y'_2(x_i) & y'_1(x_i) \end{vmatrix} / \tilde{D}(x_i, x_{i+1}) = \\ &= -W(y_1, y_2, x_i) / \tilde{D}(x_i, x_{i+1}). \end{aligned}$$

Since $p(x)W(y_1, y_2, x) = C$, where C is a constant, we see that (3.36)

holds. Hence we have

$$c_i = p(x_{i-1})\Delta_i'(x_{i-1}+0), \quad i = 1, \dots, n,$$

where $\Delta_n(t)$ is uniquely defined by

$$L\Delta_n = 0, \quad \Delta_n(x_{n-1}) = 0, \quad \Delta_n(1) = 1.$$

If $a_{12} > 0$ consider equation (3.7) and put

$$(3.37) \quad \begin{cases} d_0 = \frac{a(x_1)}{a(0)D(0, x_1)}, \\ c_1 = -\frac{1}{D(0, x_1)}. \end{cases}$$

Since $Ly = f(t, y)$ (3.7) can be written as

$$(3.38) \quad d_0 y(0) + c_1 y(x_1) = \int_0^{x_1} \Delta_0(t) Ly(t) dt + \frac{c}{a(0)B_1(b)}.$$

$\Delta_0(t)$ is uniquely defined by

$$(3.39) \quad \Delta_0(t) = \begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x_1) & y_1(x_1) \end{vmatrix} / \begin{vmatrix} y_2(0) & y_1(0) \\ y_2(x_1) & y_1(x_1) \end{vmatrix}, \quad 0 \leq t \leq x_1.$$

For a general $y \in C^2(\Omega)$ we obtain by integration by parts

$$\begin{aligned} & \int_0^{x_1} \Delta_0(t) [(p(t)y'(t))' + q(t)y(t)] dt = \\ & = p(0) [\Delta_0'(0+0)y(0) - \Delta_0(0+0)y'(0)] - \Delta_0'(x_1-0)p(x_1)y(x_1). \end{aligned}$$

Hence we have

$$(3.40) \quad L\Delta_0 = p(0) [\Delta_0'(0+0)\delta + \delta'] - \Delta_0'(x_1-0)p(x_1)\delta_{x_1}.$$

Consider the subset $B_1(c)C^2(\Omega) = \{y \in C^2(\Omega) \mid B_1 y = c\}$ of $C^2(\Omega)$.

Then for $y \in B_1(c)C^2(\Omega)$ we have

$$y'(0) = (a_{11}y(0) - c)/a_{12}$$

and

$$\begin{aligned} \langle L\Delta_0, y \rangle &= p(0) \left[\Delta'_0(0+0)y(0) - \frac{a_{11}y(0) - c}{a_{12}} \right] - \\ &- \Delta'_0(x_1-0)p(x_1)y(x_1) = \langle \Delta_0, Ly \rangle. \end{aligned}$$

By (3.38) we must have

$$\begin{cases} d_0 = p(0) \left[\Delta'_0(0+0) - \frac{a_{11}}{a_{12}} \right], \\ c_1 = - \Delta'_0(x_1-0)p(x_1). \end{cases}$$

Hence equation (3.38) can be written as

$$(3.41) \quad \langle d_0 \delta_0 + c_1 \delta_{x_1}, y \rangle = \langle \Delta_0, Ly \rangle - \frac{p(0)c}{a_{12}},$$

where by (2.6)

$$\frac{p(0)c}{a_{12}} = - \frac{c}{a(0)B_1(b)}.$$

As above $\Delta_0(t)$ is uniquely defined by

$$L\Delta_0(t) = 0, \quad 0 < t < x_1, \quad \Delta_0(0) = 1, \quad \Delta_0(x_1) = 0,$$

and hence also d_0 and c_1 . We have shown the following theorem.

Theorem 3.9

The function $\Delta_0(t)$, $0 \leq t \leq x_1 < 1$, and the constants d_0, c_1 of equation (3.38) are uniquely defined by

$$(3.42) \quad L\Delta_0(t) = 0, \quad 0 < t < x_1, \quad \Delta_0(0) = 1, \quad \Delta_0(x_1) = 0,$$

and

$$(3.43) \quad \begin{cases} d_0 = p(0) \left[\Delta'_0(0+0) - \frac{a_{11}}{a_{12}} \right], \\ c_1 = - p(x_1) \Delta'_0(x_1-0) = p(0) \Delta'_1(0+0), \end{cases}$$

if the first eigenvalue of (3.25) is greater than zero.

If $b_{22} > 0$ consider equation (3.9) and put

$$(3.44) \quad \begin{cases} c_n = -1/D(x_{n-1}, 1), \\ d_n = \frac{b(x_{n-1})}{b(1)D(x_{n-1}, 1)}. \end{cases}$$

Since $Ly = f(t, y)$ (3.9) can be written as

$$(3.45) \quad c_n y(x_{n-1}) + d_n y(1) = \int_{x_{n-1}}^1 \Delta_n(t) Ly(t) dt + \frac{d}{b(0)B_2(a)}.$$

$\Delta_n(t)$ is uniquely defined by

$$(3.46) \quad \Delta_n(t) = \frac{\begin{vmatrix} y_2(x_{n-1}) & y_1(x_{n-1}) \\ y_2(t) & y_1(t) \end{vmatrix}}{\begin{vmatrix} y_2(x_{n-1}) & y_1(x_{n-1}) \\ y_2(1) & y_1(1) \end{vmatrix}}, \quad x_{n-1} \leq t \leq 1.$$

For a general $y \in C^2(\Omega)$ we obtain by integration by parts

$$\begin{aligned} \int_{x_{n-1}}^1 \Delta_n(t) Ly(t) dt &= -p(1) [\Delta_n'(1-0)y(1) - \Delta_n(1-0)y'(1)] + \\ &+ p(x_{n-1}) \Delta_n'(x_{n-1}+0)y(x_{n-1}). \end{aligned}$$

Hence we have

$$(3.47) \quad L\Delta_n = -p(1) [\Delta_n'(1-0)\delta_1' + \delta_1'] + p(x_{n-1}) \Delta_n'(x_{n-1}+0)\delta_{x_{n-1}}.$$

Consider the subset $B_2(d)C^2(\Omega) = \{y \in C^2(\Omega) \mid B_2 y = d\}$. Then

for $y \in B_2(d)C^2(\Omega)$ we have

$$y'(1) = (d - b_{21}y(1))/b_{22}$$

and

$$\begin{aligned} \langle L\Delta_n, y \rangle &= -p(1) [\Delta_n'(1-0)y(1) - y'(1)] + \\ &+ p(x_{n-1}) \Delta_n'(x_{n-1}+0)y(x_{n-1}) = \langle \Delta_n, Ly \rangle. \end{aligned}$$

By (3.45) we must have

$$\begin{cases} c_n = p(x_{n-1})\Delta'_n(x_{n-1}+0), \\ d_n = -p(1)\left[\Delta'_n(1-0) + \frac{b_{21}}{b_{22}}\right]. \end{cases}$$

Hence equation (3.45) can be written as

$$(3.48) \quad \langle c_n \delta_{x_{n-1}} + d_n \delta_{1,y} \rangle = \langle \Delta_n, Ly \rangle - \frac{p(1)d}{b_{22}},$$

where by (2.7)

$$\frac{p(1)d}{b_{22}} = -\frac{d}{b(1)B_2(a)}.$$

As above $\Delta_n(t)$ is uniquely defined by

$$L\Delta_n(t) = 0, \quad x_{n-1} < t < 1, \quad \Delta_n(x_{n-1}) = 0, \quad \Delta_n(1) = 1,$$

and hence also c_n and d_n . We have shown the following theorem.

Theorem 3.10

The function $\Delta_n(t)$, $0 \leq x_{n-1} \leq t < 1$, and the constants c_n and d_n of equation (3.45) are uniquely defined by

$$(3.49) \quad L\Delta_n(t) = 0, \quad x_{n-1} < t < 1, \quad \Delta_n(x_{n-1}) = 0, \quad \Delta_n(1) = 1,$$

and,

$$(3.50) \quad \begin{cases} c_n = p(x_n)\Delta'_n(x_{n-1}+0) = -p(1)\Delta'_{n-1}(1-0), \\ d_n = -p(1)\left[\Delta'_n(1-0) + \frac{b_{21}}{b_{22}}\right], \end{cases}$$

if the first eigenvalue of (3.25) is greater than zero.

Theorems 3.8, 3.9, and 3.10 give an alternative definition of the determinant formulation of the boundary value problem (2.1).

Equation (3.5) is well defined by Theorem 3.8 if the boundary value problems in (3.34) are uniquely solvable. If $a_{12} > 0$ then equation (3.7) is well defined by Theorem 3.9 if the boundary

value problem (3.43) has a unique solution. If $b_{22} > 0$ then equation (3.9) is well define if the boundary value problem (3.49) has a unique solution. The preceding boundary value problems have unique solutions if and only if $\tilde{D}(x_{i-1}, x_i) \neq 0$ for $i = 1, \dots, n$. By Theorem 3.1 we always have $\tilde{D}(x_{i-1}, x_i) \neq 0$ if $x_i - x_{i-1}$ is sufficiently small. Thus also if zero is an eigenvalue of $(L, \mathcal{B})$ (i.e. Green's function for $(L, \mathcal{B})$ does not exists) the determiniant formulation exists if $x_i - x_{i-1}$, $i = 1, \dots, n$, are sufficiently small.

Example 3.5

Consider the boundary value problem

$$(3.51) \quad \begin{cases} y'' + \pi^2 y = f(t, y(t)), & 0 < t < 1, \\ y(0) = y(1) = 0, \end{cases}$$

where $\lambda = \pi^2$ is an eigenvalue of (3.25) with $L = \frac{d^2}{dt^2}$, and the grid $x_i = i/4$, $i = 0, 1, 2, 3, 4$. Let $y_1(t) = \sin(\pi t)$, $y_2(t) = \cos(\pi t)$.

By Theorem 3.8 we have

$$\begin{cases} c_i = \sqrt{2}\pi, & i = 1, 2, 3, 4 \\ d_i = -2\pi, & i = 1, 2, 3, \end{cases}$$

and

$$\begin{aligned} \Delta_1(t) &= \begin{cases} \sqrt{2} \sin(\pi t), & 0 \leq t \leq \frac{1}{4}, \\ \sqrt{2} \cos(\pi t), & \frac{1}{4} \leq t \leq \frac{1}{2}, \end{cases} \\ \Delta_2(t) &= \begin{cases} \sin(\pi t) - \cos(\pi t), & \frac{1}{4} \leq t \leq \frac{1}{2}, \\ \sin(\pi t) + \cos(\pi t), & \frac{1}{2} \leq t \leq \frac{3}{4}, \end{cases} \\ \Delta_3(t) &= \begin{cases} -\sqrt{2} \cos(\pi t), & \frac{1}{2} \leq t \leq \frac{3}{4}, \\ \sqrt{2} \sin(\pi t), & \frac{3}{4} \leq t \leq 1. \end{cases} \end{aligned}$$

Hence the determinant formulation of the boundary value problem (3.51) exists and we have

$$\sqrt{2}\pi y(x_{i-1}) - 2\pi y(x_i) + \sqrt{2}\pi y(x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) f(t, y(t)) dt$$

for $i = 1, 2, 3$, where $y(x_0) = y(x_4) = 0$. The matrix of the left hand side is

$$T = \begin{bmatrix} -2\pi & \sqrt{2}\pi & 0 \\ \sqrt{2}\pi & -2\pi & \sqrt{2}\pi \\ 0 & \sqrt{2}\pi & -2\pi \end{bmatrix}.$$

Since Green's function for the boundary value problem (3.51) does not exist we must have $\det T = 0$, which can be confirmed by direct computation. Hence we have a boundary value problem for which the integral equation formulation does not exist but the determinant formulation exists.

It may look like this example contradict Theorem 3.3 but if we note the influence of $a_0 = b_n = 0$, in Theorem 3.3, on the existence of T^{-1} we see why this is not the case.

We will now obtain the limit of equation (3.2) when $x_1, x_3 \rightarrow x_2$, and also of (3.3) when $x_1 \rightarrow 0$ and of (3.4) when $x_3 \rightarrow 1$. By determinant manipulation equation (3.2) can be written as

$$(3.52) \quad \begin{vmatrix} b(x_1) & a(x_1) & y(x_1) - \int_{x_1}^{x_3} G(x_1, t) f dt \\ b(x_1, x_2) & a(x_1, x_2) & y(x_1, x_2) - \int_{x_1}^{x_3} G(x_1, x_2; t) f dt \\ b(x_1, x_2, x_3) & a(x_1, x_2, x_3) & y(x_1, x_2, x_3) - \int_{x_1}^{x_3} G(x_1, x_2, x_3; t) f dt \end{vmatrix} = 0,$$

where $y(x_1, x_2)$ and $y(x_1, x_2, x_3)$ denote divided differences of y .

Let $x_1 \rightarrow x_2$ and $x_3 \rightarrow x_2$, then we get

$$\begin{aligned}
b(x_1, x_2) &\rightarrow b'(x_2), & b(x_1, x_2, x_3) &\rightarrow \frac{1}{2}b''(x_2), \\
a(x_1, x_2) &\rightarrow a'(x_2), & a(x_1, x_2, x_3) &\rightarrow \frac{1}{2}a''(x_2), \\
y(x_1, x_2) &\rightarrow y'(x_2), & y(x_1, x_2, x_3) &\rightarrow \frac{1}{2}y''(x_2).
\end{aligned}$$

For the integrals in the first and second row of (3.52) we get

$$\int_{x_1}^{x_3} G(x_1, t) f(t, y(t)) dt \rightarrow 0, \quad \text{when } x_1, x_3 \rightarrow x_2,$$

$$\int_{x_1}^{x_3} G(x_1, x_2; t) f(t, y(t)) dt \rightarrow 0, \quad \text{when } x_1, x_3 \rightarrow x_2.$$

For the integral in the third row a simple Taylor argument shows that

$$\begin{aligned}
&\int_{x_1}^{x_3} G(x_1, x_2, x_3; t) f(t, y(t)) dt \rightarrow \\
&\rightarrow [a(x_2)b'(x_2) - a'(x_2)b(x_2)] \frac{f(x_2, y(x_2))}{2},
\end{aligned}$$

when $x_1, x_3 \rightarrow x_2$. Hence when $x_1, x_3 \rightarrow x_2$ equation (3.52) converges to

$$(3.53) \quad \begin{vmatrix} b(x_2) & a(x_2) & y(x_2) \\ b'(x_2) & a'(x_2) & y'(x_2) \\ b''(x_2) & a''(x_2) & y''(x_2) - [a(x_2)b'(x_2) - a'(x_2)b(x_2)] f \end{vmatrix} = 0.$$

By expanding the determinant after the third column we get

$$\begin{aligned}
(3.54) \quad &(a'b - ab')y'' - (a''b - ab'')y' + (a''b' - a'b'')y = \\
&= - (ab' - a'b)^2 f(t, y).
\end{aligned}$$

Now $ab' - a'b = 1/p$, $ab'' - a''b = -p'/p^2$, and since $La = Lb = 0$ we get $p(a''a' - a'b'') = -q(ab' - a'b) = -q/p$. Hence (3.54) can be written as

$$-\frac{1}{p} y'' - \frac{p'}{p^2} y' - \frac{q}{p^2} y = -\frac{1}{p^2} f(t, y),$$

and if we multiply this equation with $-p^2$ we recognize the original equation $py'' + p'y' + qy = f(t, y)$.

To derive the boundary conditions consider equation (3.3), which we rewrite as

$$(3.55) \quad \begin{vmatrix} a(0) & y(0) - \int_0^{x_1} G(0, t) f(t, y(t)) dt - \frac{cb(0)}{B_1(b)} \\ a(0, x_1) & y(0, x_1) - \int_0^{x_1} G(0, x_1; t) f(t, y(t)) dt - \frac{cb(0, x_1)}{B_1(b)} \end{vmatrix} = 0.$$

Let $x_1 \rightarrow 0$ then we get

$$\begin{vmatrix} a(0) & y(0) - cb(0)/B_1(b) \\ a'(0) & y'(0) - cb'(0)/B_1(b) \end{vmatrix} = 0,$$

or

$$(3.56) \quad a(0)y'(0) - a'(0)y(0) = \frac{c}{B_1(b)} [a(0)b'(0) - a'(0)b(0)] = \\ = \frac{c}{p(0)B_1(b)}.$$

By (2.6) we have

$$p(0)B_1(b)a(0) = -a_{12} \quad \text{and} \quad p(0)B_1(b)a'(0) = -a_{11}.$$

Hence (3.56) can be written as

$$a_{11}y(0) - a_{12}y'(0) = c$$

which is the boundary condition of (2.1) at the point $x = 0$.

Now consider the other endpoint. Equation (3.4) can be written as

$$(3.57) \quad \begin{vmatrix} b(x_3) & y(x_3) - \int_{x_3}^1 G(x_3, t) f(t, y(t)) dt - \frac{da(x_3)}{B_2(a)} \\ b(x_3, 1) & y(x_3, 1) - \int_{x_3}^1 G(x_3, 1; t) f(t, y(t)) dt - \frac{da(x_3, 1)}{B_2(a)} \end{vmatrix} = 0.$$

Let $x_3 \rightarrow 1$ then we get

$$\begin{vmatrix} b(1) & y(1) - da(1)/B_2(a) \\ b'(1) & y'(1) - da'(1)/B_2(a) \end{vmatrix} = 0,$$

or

$$(3.58) \quad b(1)y'(1) - b'(1)y(1) = \frac{d}{b_2(a)} [a'(1)b(1) - a(1)b'(1)] = \\ = - \frac{d}{p(1)B_2(a)}.$$

By (2.7) we have

$$-p(1)B_2(a)b(1) = b_{22} \quad \text{and} \quad p(1)B_2(a)b'(1) = b_{21}.$$

Hence (3.58) can be written as

$$b_{21}y(1) + b_{22}y'(1) = d$$

which is the boundary condition of (2.1) at the point $x = 1$.

Thus (3.3) and (3.4) give back the boundary conditions of equation (2.1).

We will now consider the determinant formulation of the boundary value problem (2.26). By (2.32) and (2.33) we have

$$(3.59) \quad y(x_1) - b(x_1) \int_0^{x_1} c(t) f(t, y(t)) dt - a(x_1) \int_{x_3}^1 d(t) f(t, y(t)) dt - \\ - \int_{x_1}^{x_3} G(x_1, t) f(t, y(t)) dt - \frac{cb(x_1)}{B_2(b)} - \frac{da(x_1)}{B_1(a)} = 0,$$

for $i = 1, 2, 3$, where $0 \leq x_1 < x_2 < x_3 \leq 1$. Hence the determinant formulation of the integral equation (2.33) is given by (3.2), (3.3), and (3.4) with $G(x, t)$ defined by (2.32). Since a and b also satisfy the homogeneous equation

$$\frac{1}{p_0} \exp\left(\int_0^x \frac{p_1}{p_0} dt\right) Ly = 0$$

we can apply Theorem 3.1 and 3.7 with p and q defined by (2.29). Hence consider the grid $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$. Put

$$D(x_i, x_{i+1}) = \begin{vmatrix} b(x_i) & a(x_i) \\ b(x_{i+1}) & a(x_{i+1}) \end{vmatrix}$$

and assume that

$$D(x_i, x_{i+1}) \neq 0, \quad \text{for } i = 0, 1, \dots, n-1.$$

Then by determinant manipulation equation (3.2), with $G(x, t)$ defined by (2.32), can be written as

$$(3.60) \quad \frac{-y(x_{i-1})}{D(x_{i-1}, x_i)} + \frac{D(x_{i-1}, x_{i+1})}{D(x_{i-1}, x_i)D(x_i, x_{i+1})} y(x_i) + \frac{-y(x_{i+1})}{D(x_i, x_{i+1})} =$$

$$= \int_{x_{i-1}}^{x_{i+1}} T_i(t) f(t, y(t)) dt, \quad i = 1, \dots, n-1,$$

where

$$(3.61) \quad T_i(t) = \begin{cases} \frac{\begin{vmatrix} b(x_{i-1}) & a(x_{i-1}) \\ d(t) & c(t) \end{vmatrix}}{D(x_{i-1}, x_i)}, & x_{i-1} \leq t \leq x_i, \\ \frac{\begin{vmatrix} d(t) & c(t) \\ b(x_{i+1}) & a(x_{i+1}) \end{vmatrix}}{D(x_i, x_{i+1})}, & x_i \leq t \leq x_{i+1}. \end{cases}$$

By (2.31) we have

$$(3.62) \quad T_i(x_{i-1}) = T_i(x_{i+1}) = 0$$

and

$$(3.63) \quad T_i(x_i) = \frac{1}{p_0(x_i)W(a, b, x_i)}$$

for $i = 1, \dots, n-1$.

If $a_{12} > 0$ then $a(0) \neq 0$ and equation (3.3), with $G(x, t)$ defined by (2.32), can be written as

$$(3.64) \quad \frac{a(x_1)}{a(0)D(0, x_1)} y(0) + \frac{-y(x_1)}{D(0, x_1)} = \int_0^{x_1} T_0(t)f(t, y(t))dt + \frac{c}{a(0)B_1(b)},$$

where

$$(3.65) \quad T_0(t) = \frac{\begin{vmatrix} d(t) & c(t) \\ b(x_1) & a(x_1) \end{vmatrix}}{D(0, x_1)}, \quad 0 \leq t \leq x_1.$$

If $b_{22} > 0$ then $b(1) \neq 0$ and equation (3.4), with $G(x, t)$ defined by (2.32), can be written as

$$(3.66) \quad \frac{-y(x_{n-1})}{D(x_{n-1}, 1)} + \frac{b(x_{n-1})}{b(1)D(x_{n-1}, 1)} y(1) = \int_{x_{n-1}}^1 T_n(t)f(t, y(t))dt + \\ + \frac{d}{b(1)B_2(a)},$$

where

$$(3.67) \quad T_n(t) = \frac{\begin{vmatrix} b(x_{n-1}) & a(x_{n-1}) \\ d(t) & c(t) \end{vmatrix}}{D(x_{n-1}, 1)}, \quad x_{n-1} \leq t \leq 1.$$

We have

$$(3.68) \quad \begin{cases} T_0(x_1) = T_n(x_{n-1}) = 0, \\ T_0(0) = \frac{1}{p(0)W(a, b, 0)}, \quad T_n(1) = \frac{1}{p(1)W(a, b, 1)}. \end{cases}$$

We have in this chapter considered the determinant formulation of an integral equation of the form

$$(3.69) \quad y(x) = \int_0^1 G(x,t)f(t,y(t))dt + h(x), \quad 0 \leq x \leq 1,$$

where

$$G(x,t) = \begin{cases} b(x)a(t), & x \geq t, \\ a(x)b(t), & x \leq t. \end{cases}$$

The kernel G is called semidegenerate. Generally a kernel K is called semidegenerate if it has the form

$$(3.70) \quad K(x,t) = \begin{cases} \sum_{i=1}^M a_i(x)b_i(t), & x > t, \\ \sum_{j=1}^N c_j(x)d_j(t), & x < t. \end{cases}$$

All the kernels considered in chapter 2 is of the form (3.70)

with $M = N = 1$.

Determinant formulation of integral equations of the form

$$(3.71) \quad y(x) = \int_0^1 K(x,t)f(t,y(t))dt + h(x), \quad 0 \leq x \leq 1,$$

where K is given by (3.70), has been given in Aulin [4]. This formulation can also be extended to nonlinear equations of Urysohn type

$$(3.72) \quad y(x) = \int_0^1 K(x,t,y(t))dt + h(x), \quad 0 \leq x \leq 1,$$

where

$$(3.73) \quad K(x,t,y(t)) = \begin{cases} \sum_{i=1}^M a_i(x)b_i(t,y(t)), & x > t, \\ \sum_{j=1}^N c_j(x)d_j(t,y(t)), & x < t. \end{cases}$$

4. DISCRETIZATION OF TWO-POINT BOUNDARY VALUE PROBLEMS

We will first consider the boundary value problem (2.1) and its determinant formulation given by (3.5), (3.7), and (3.9). We assume throughout this chapter that Theorem 3.7 holds.

Consider the grid $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$. Approximation of the integrals in the right hand sides of (3.5), (3.7), and (3.9) by the trapezoidal rule at the points x_i , $i = 0, \dots, n$, gives the following discretization of the determinant formulation of (2.1) ($y_i \approx y(x_i)$)

$$(4.1) \quad \left\{ \begin{array}{l} \frac{a(x_1)}{a(0)D(0, x_1)} y_0 + \frac{-y_1}{D(0, x_1)} = w_0 f(0, y_0) + \frac{c}{a(0)B_1(b)}, \quad \text{if } a_{12} > 0, \\ \\ \frac{-y_{i-1}}{D(x_{i-1}, x_i)} + \frac{D(x_{i-1}, x_{i+1})}{D(x_{i-1}, x_i)D(x_i, x_{i+1})} y_i + \frac{-y_{i+1}}{D(x_i, x_{i+1})} = \\ \\ = w_i f(x_i, y_i), \quad i = 1, \dots, n-1, \\ \\ \frac{-y_{n-1}}{D(x_{n-1}, 1)} + \frac{b(x_{n-1})}{b(1)D(x_{n-1}, 1)} y_n = w_n f(1, y_n) + \frac{d}{b(1)B_2(a)}, \quad \text{if } b_{22} > 0, \end{array} \right.$$

where

$$(4.2) \quad w_i = \begin{cases} \frac{x_1}{2}, & \text{when } i = 0, \\ \frac{x_{i+1} - x_{i-1}}{2}, & \text{when } i = 1, \dots, n-1, \\ \frac{1 - x_{n-1}}{2}, & \text{when } i = n. \end{cases}$$

Approximation of the integral equation (2.5) by the trapezoidal rule on the same grid gives the sum equation ($y_i \approx y(x_i)$)

$$(4.3) \quad y_i = \sum_{j=0}^n w_j G(x_i, x_j) f(x_j, y_j) + \frac{cb(x_i)}{B_1(b)} + \frac{da(x_i)}{B_2(a)},$$

for $i = 0, \dots, n$. Since we have used the compound trapezoidal rule

we get, by exactly the same method as used for the integral equation (2.5), the following determinant formulation of the sum equation (4.3)

$$(4.4) \quad \begin{vmatrix} b(x_{i-1}) & a(x_{i-1}) & y_{i-1} - \sum_{j=i-1}^{i+1} w_{i,j} G(x_{i-1}, x_j) f(x_j, y_j) \\ b(x_i) & a(x_i) & y_i - \sum_{j=i-1}^{i+1} w_{i,j} G(x_i, x_j) f(x_j, y_j) \\ b(x_{i+1}) & a(x_{i+1}) & y_{i+1} - \sum_{j=i-1}^{i+1} w_{i,j} G(x_{i+1}, x_j) f(x_j, y_j) \end{vmatrix} = 0,$$

for $i = 1, \dots, n-1$, where $w_{i,i-1} = (x_i - x_{i-1})/2$, $w_{i,i} = w_i$, and $w_{i,i+1} = (x_{i+1} - x_i)/2$,

$$(4.5) \quad \begin{vmatrix} a(0) & y_0 - \sum_{j=0}^1 w_{0,j} G(0, x_j) f(x_j, y_j) - \frac{cb(0)}{B_1(b)} \\ a(x_1) & y_1 - \sum_{j=0}^1 w_{0,j} G(x_1, x_j) f(x_j, y_j) - \frac{cb(x_1)}{B_1(b)} \end{vmatrix} = 0,$$

where $w_{0,0} = w_{0,1} = w_0$,

$$(4.6) \quad \begin{vmatrix} b(x_{n-1}) & y_{n-1} - \sum_{j=n-1}^n w_{n,j} G(x_{n-1}, x_j) f(x_j, y_j) - \frac{da(x_{n-1})}{B_2(a)} \\ b(1) & y_n - \sum_{j=n-1}^n w_{n,j} G(1, x_j) f(x_j, y_j) - \frac{da(1)}{B_2(a)} \end{vmatrix} = 0,$$

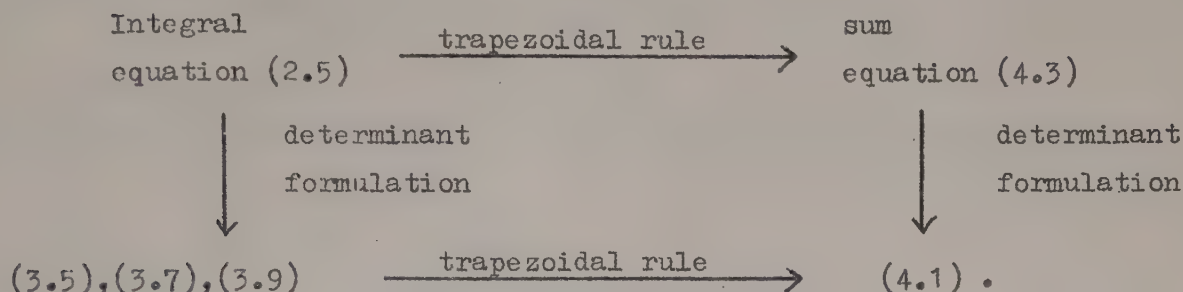
where $w_{n,n-1} = w_{n,n} = w_n$.

By determinant manipulation it is easy to see that the system defined by (4.4), (4.5), and (4.6) is exactly the same system as (4.1) (with obvious modifications when $a_{12} = 0$ or $b_{22} = 0$). We have shown the following theorem.

Theorem 4.1

For the integral equation formulation of the boundary value problem (2.1) the trapezoidal rule commutes with the determinant formulation of the problem.

Hence we have the following commutative diagram



In the approximation of the integral

$$\int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) f(t, y(t)) dt$$

by the trapezoidal rule we used the function values

$F_i(t) = \Delta_i(t) f(t, y(t))$ at the points $t = x_{i-1}, x_i, x_{i+1}$. However, it is possible to derive approximation rules that make use of other sorts of functional information. If we also make use of the derivative values at $t = x_{i-1}, x_i, x_{i+1}$ of F_i we can use the trapezoidal rule with "end correction" to approximate the integral.

We have

$$\begin{aligned}
 \int_{x_{i-1}}^{x_i} F_i(t) dt &= \frac{x_i - x_{i-1}}{2} [F_i(x_{i-1}) + F_i(x_i)] + \\
 &+ \frac{(x_i - x_{i-1})^2}{12} [F'_i(x_{i-1}) - F'_i(x_i)] + \frac{(x_i - x_{i-1})^5}{720} F_i^{(4)}(\xi),
 \end{aligned}$$

$x_{i-1} < \xi < x_i$, if F_i is a C^4 -function in the interval (x_{i-1}, x_i)

(see Davis and Rabinowitz [11] p. 105). We get

$$\begin{aligned}
 \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) f(t, y(t)) dt &= \int_{x_{i-1}}^{x_i} \Delta_i(t) f(t, y(t)) dt + \int_{x_i}^{x_{i+1}} \Delta_i(t) f(t, y(t)) dt \approx \\
 &\approx \frac{x_{i+1} - x_{i-1}}{2} f(x_i, y(x_i)) + \frac{(x_i - x_{i-1})^2}{12} \Delta'_i(x_{i-1}+0) f(x_{i-1}, y(x_{i-1})) + \\
 &+ \left[\frac{(x_{i+1} - x_i)^2}{12} \Delta'_i(x_i+0) - \frac{(x_i - x_{i-1})^2}{12} \Delta'_i(x_i-0) \right] f(x_i, y(x_i)) + \\
 &+ \left[\frac{(x_{i+1} - x_i)^2}{12} - \frac{(x_i - x_{i-1})^2}{12} \right] \left(\frac{d}{dt} f(t, y(t)) \Big|_{t=x_i} \right) - \\
 &- \frac{(x_{i+1} - x_i)^2}{12} \Delta'_i(x_{i+1}-0) f(x_{i+1}, y(x_{i+1})).
 \end{aligned}$$

If $x_{i+1} - x_i = x_i - x_{i-1} = h = 1/n$ we get

$$\begin{aligned}
 \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) f(t, y(t)) dt &\approx hf(x_i, y(x_i)) + \frac{h^2}{12} \left[\frac{-f(x_{i-1}, y(x_{i-1}))}{p(x_{i-1})D(x_{i-1}, x_i)} + \right. \\
 &+ \left. \frac{D(x_{i-1}, x_{i+1})}{p(x_i)D(x_{i-1}, x_i)D(x_i, x_{i+1})} f(x_i, y(x_i)) + \frac{-f(x_{i+1}, y(x_{i+1}))}{p(x_{i+1})D(x_i, x_{i+1})} \right].
 \end{aligned}$$

Hence for the boundary value problem (2.1), with $a_{12} = b_{22} = 0$, the approximation of (3.5) by the trapezoidal rule with "end correction" on the grid $x_i = \frac{i}{n}$, $i = 0, \dots, n$, gives the following system ($y_i \approx y(\frac{i}{n})$)

$$\begin{aligned}
 (4.7) \quad & \frac{-y_{i-1}}{D(\frac{i-1}{n}, \frac{i}{n})} + \frac{D(\frac{i-1}{n}, \frac{i+1}{n})}{D(\frac{i-1}{n}, \frac{i}{n})D(\frac{i}{n}, \frac{i+1}{n})} y_i + \frac{-y_{i+1}}{D(\frac{i}{n}, \frac{i+1}{n})} = hf(\frac{i}{n}, y_i) + \\
 & + \frac{h^2}{12} \left[\frac{-f(\frac{i-1}{n}, y_{i-1})}{p(\frac{i-1}{n})D(\frac{i-1}{n}, \frac{i}{n})} + \frac{D(\frac{i-1}{n}, \frac{i+1}{n})}{p(\frac{i}{n})D(\frac{i-1}{n}, \frac{i}{n})D(\frac{i}{n}, \frac{i+1}{n})} f(\frac{i}{n}, y_i) + \right. \\
 & \left. + \frac{-f(\frac{i+1}{n}, y_{i+1})}{p(\frac{i+1}{n})D(\frac{i}{n}, \frac{i+1}{n})} \right],
 \end{aligned}$$

for $i = 1, \dots, n-1$, $h = 1/n$.

The invers of the matrix defined by the left hand side of (4.7) is

$G = (g_{i,j})_{i,j=1}^{n-1}$, where

$$g_{i,j} = \begin{cases} b(\frac{i}{n})a(\frac{j}{n}), & i \geq j, \\ a(\frac{i}{n})b(\frac{j}{n}), & i \leq j. \end{cases}$$

Hence we have

$$\begin{aligned}
 (4.8) \quad y_i &= \sum_{j=1}^{n-1} hg_{i,j} f(\frac{j}{n}, y_j) + g_{i,1} \frac{y_0}{D(0, \frac{1}{n})} + \frac{y_n}{D(\frac{n-1}{n}, 1)} g_{i,n-1} + \\
 & + \frac{h^2}{12} \sum_{j=1}^{n-1} g_{i,j} \left[\frac{-f(\frac{j-1}{n}, y_{j-1})}{p(\frac{j-1}{n})D(\frac{j-1}{n}, \frac{j}{n})} + \frac{D(\frac{j-1}{n}, \frac{j+1}{n})}{p(\frac{j}{n})D(\frac{j-1}{n}, \frac{j}{n})D(\frac{j}{n}, \frac{j+1}{n})} f(\frac{j}{n}, y_j) + \right. \\
 & \left. + \frac{-f(\frac{j+1}{n}, y_{j+1})}{p(\frac{j+1}{n})D(\frac{j}{n}, \frac{j+1}{n})} \right],
 \end{aligned}$$

for $i = 1, \dots, n-1$, $h = 1/n$. The last sum can be written as

$$\begin{aligned}
& - \varepsilon_{i,1} \frac{f(0, y_0)}{p(0)D(0, \frac{1}{n})} + \sum_{j=1}^{n-2} \varepsilon_{i,j+1} \frac{-f(\frac{j}{n}, y_j)}{p(\frac{j}{n})D(\frac{j}{n}, \frac{j+1}{n})} + \\
& + \sum_{j=1}^{n-1} \varepsilon_{i,j} \frac{D(\frac{j-1}{n}, \frac{j+1}{n})}{p(\frac{j}{n})D(\frac{j-1}{n}, \frac{j}{n})D(\frac{j}{n}, \frac{j+1}{n})} f(\frac{j}{n}, y_j) + \\
& + \sum_{j=2}^{n-1} \varepsilon_{i,j-1} \frac{-f(\frac{j}{n}, y_j)}{p(\frac{j}{n})D(\frac{j-1}{n}, \frac{j}{n})} - \varepsilon_{i,n-1} \frac{f(1, y_n)}{p(1)D(\frac{n-1}{n}, 1)} = \\
& = - \varepsilon_{i,1} \frac{f(0, y_0)}{p(0)D(0, \frac{1}{n})} + \left[\frac{-\varepsilon_{i,2}}{D(\frac{1}{n}, \frac{2}{n})} + \varepsilon_{i,1} \frac{D(0, \frac{2}{n})}{D(0, \frac{1}{n})D(\frac{1}{n}, \frac{2}{n})} \right] \frac{f(\frac{1}{n}, y_1)}{p(\frac{1}{n})} + \\
& + \sum_{j=2}^{n-2} \left[\frac{-\varepsilon_{i,j+1}}{D(\frac{j}{n}, \frac{j+1}{n})} - \varepsilon_{i,j} \frac{D(\frac{j-1}{n}, \frac{j+1}{n})}{D(\frac{j-1}{n}, \frac{j}{n})D(\frac{j}{n}, \frac{j+1}{n})} + \frac{-\varepsilon_{i,j-1}}{D(\frac{j-1}{n}, \frac{j}{n})} \right] \frac{f(\frac{j}{n}, y_j)}{p(\frac{j}{n})} + \\
& + \left[\varepsilon_{i,n-1} \frac{D(\frac{n-2}{n}, 1)}{D(\frac{n-2}{n}, \frac{n-1}{n})D(\frac{n-1}{n}, 1)} + \frac{-\varepsilon_{i,n-2}}{D(\frac{n-2}{n}, \frac{n-1}{n})} \right] \frac{f(\frac{n-1}{n}, y_{n-1})}{p(\frac{n-1}{n})} - \\
& - \varepsilon_{i,n-1} \frac{f(1, y_n)}{p(1)D(\frac{n-1}{n}, 1)},
\end{aligned}$$

for $i = 1, \dots, n-1$. We have

$$\begin{aligned}
& - \varepsilon_{i,j+1} D(\frac{j-1}{n}, \frac{j}{n}) + \varepsilon_{i,j} D(\frac{j-1}{n}, \frac{j+1}{n}) - \varepsilon_{i,j-1} D(\frac{j}{n}, \frac{j+1}{n}) = \\
& = - \begin{vmatrix} b(\frac{j-1}{n}) & a(\frac{j-1}{n}) & \varepsilon_{i,j-1} \\ b(\frac{j}{n}) & a(\frac{j}{n}) & \varepsilon_{i,j} \\ b(\frac{j+1}{n}) & a(\frac{j+1}{n}) & \varepsilon_{i,j+1} \end{vmatrix} = - D_{i,j},
\end{aligned}$$

for $j = 2, \dots, n-2$, and since $\varepsilon_{i,0} = \varepsilon_{i,n} = 0$ also for $j = 1$ and $j = n-1$. Hence

$$D_{i,j} = \begin{cases} -D(\frac{i-1}{n}, \frac{i}{n})D(\frac{i}{n}, \frac{i+1}{n}), & i = j, \\ 0, & i \neq j. \end{cases}$$

Now equation (4.8) takes the form

$$(4.9) \quad y_i = h \sum_{j=1}^{n-1} g_{i,j} f(\frac{j}{n}, y_j) + \frac{b(\frac{i}{n})c}{B_1(b)} + \frac{a(\frac{i}{n})d}{B_2(a)} + \\ + \frac{h^2}{12} \left[-g_{i,j} \frac{f(0, y_0)}{p(0)D(0, \frac{1}{n})} + \frac{f(\frac{i}{n}, y_i)}{p(\frac{i}{n})} - g_{i,n-1} \frac{f(1, y_n)}{p(1)D(\frac{n-1}{n}, 1)} \right],$$

since $a_{11}y(0) = c$, $b_{21}y(1) = d$, $a(0) = b(1) = 0$. We also have

$$g_{i,1} \frac{f(0, y_0)}{p(0)D(0, \frac{1}{n})} = \frac{b(\frac{i}{n})a(\frac{1}{n})f(0, y_0)}{p(0)[b(0)a(\frac{1}{n}) - a(0)b(\frac{1}{n})]} = \frac{b(\frac{i}{n})}{p(0)b(0)} f(0, y_0),$$

since $a(0) = 0$ and

$$g_{i,n-1} \frac{f(1, y_n)}{p(1)D(\frac{n-1}{n}, 1)} = \frac{a(\frac{i}{n})}{p(1)a(1)} f(1, y_n),$$

since $b(1) = 0$. Hence equation (4.8) can be written as

$$(4.10) \quad y_i = h \sum_{j=1}^{n-1} g_{i,j} f(\frac{j}{n}, y_j) + \frac{b(\frac{i}{n})c}{B_1(b)} + \frac{a(\frac{i}{n})d}{B_2(a)} + \\ + \frac{h^2}{12} \left[-\frac{b(\frac{i}{n})}{p(0)b(0)} f(0, y_0) + \frac{f(\frac{i}{n}, y_i)}{p(\frac{i}{n})} - \frac{a(\frac{i}{n})}{p(1)a(1)} f(1, y_n) \right],$$

for $i = 1, \dots, n-1$.

Now consider the integral equation (2.5) with $a_{12} = b_{22} = 0$.

Approximation of the integral in (2.5) by the compound trapezoidal rule with "end correction" on the grid $x_i = \frac{i}{n}$, $i = 0, \dots, n$, gives

$$\begin{aligned}
y\left(\frac{i}{n}\right) \approx & h \sum_{j=1}^{i-1} g_{i,j} f\left(\frac{j}{n}, y\left(\frac{j}{n}\right)\right) + \frac{h}{2} g_{i,i} f\left(\frac{i}{n}, y\left(\frac{i}{n}\right)\right) + \\
& + \frac{h^2}{12} b\left(\frac{i}{n}\right) \left[a'(0) f(0, y(0)) + a(0) \frac{d}{dt} f(t, y(t)) \Big|_{t=0} - \right. \\
& - a'\left(\frac{i}{n}\right) f\left(\frac{i}{n}, y\left(\frac{i}{n}\right)\right) - a\left(\frac{i}{n}\right) \frac{d}{dt} f(t, y(t)) \Big|_{t=i/n} \Big] + \\
& + \frac{h}{2} g_{i,i} f\left(\frac{i}{n}, y\left(\frac{i}{n}\right)\right) + h \sum_{j=i+1}^{n-1} g_{i,j} f\left(\frac{j}{n}, y\left(\frac{j}{n}\right)\right) + \\
& + \frac{h^2}{12} a\left(\frac{i}{n}\right) \left[b'\left(\frac{i}{n}\right) f\left(\frac{i}{n}, y\left(\frac{i}{n}\right)\right) + b\left(\frac{i}{n}\right) \frac{d}{dt} f(t, y(t)) \Big|_{t=i/n} - \right. \\
& - b'(1) f(1, y(1)) - b(1) \frac{d}{dt} f(t, y(t)) \Big|_{t=1} \Big] + \\
& + \frac{b\left(\frac{i}{n}\right)c}{B_1(b)} + \frac{a\left(\frac{i}{n}\right)d}{B_2(a)}.
\end{aligned}$$

Hence with $y\left(\frac{i}{n}\right) := y_i$ we get the system (4.10) since

$$-\frac{1}{p(0)b(0)} = -\frac{a(0)b'(0) - a'(0)b(0)}{b(0)} = a'(0),$$

and

$$\frac{1}{p(1)a(1)} = \frac{a(1)b'(1) - a'(1)b(1)}{a(1)} = b'(1).$$

Thus (4.10) is the approximation of the integral equation (2.5) by the trapezoidal rule with "end correction". We have proved the following theorem.

Theorem 4.2

For the integral equation formulation (2.5) of the boundary value problem (2.1) with $a_{12} = b_{22} = 0$ the trapezoidal rule with "end correction" commutes with the determinant formulation of the problem.

If $a_{12} > 0$ we also have the equation

$$(4.11) \quad \frac{a(x_1)}{a(0)D(0, x_1)} y(0) - \frac{y(x_1)}{D(0, x_1)} = \int_0^{x_1} \Delta_0(t) f(t, y(t)) dt + \frac{c}{a(0)B_1(b)}.$$

Approximation of the integral by the trapezoidal rule with "end correction" will give a term containing $\frac{d}{dt} f(t, y(t)) \Big|_{t=0}$ since $\Delta_0(0) = 1$. To get an $O(h^4)$ -approximation of the integral we could use Simpson's rule at the points $0, x_1/2, x_1$. However when the function f depends on y the point $x_1/2$ will introduce a new variable $y(x_1/2)$ in the system and we will need another equation. Instead we consider the determinant formulation of the boundary value problem at the points 0 and x_2 , we get

$$(4.12) \quad \frac{a(x_2)}{a(0)D(0, x_2)} y(0) - \frac{y(x_2)}{D(0, x_2)} = \int_0^{x_2} \frac{D(t, x_2)}{D(0, x_2)} f(t, y(t)) dt + \frac{c}{a(0)B_1(b)}.$$

Now with $x_1 = 1/n, x_2 = 2/n, h = 1/n$ the approximation of (4.12) by Simpson's rule gives $(y_i \approx y(\frac{i}{n}))$

$$(4.13) \quad \frac{a(\frac{2}{n})}{a(0)D(0, \frac{2}{n})} y_0 - \frac{y_2}{D(0, \frac{2}{n})} = \frac{h}{3} \left[f(0, y_0) + 4 \frac{D(\frac{1}{n}, \frac{2}{n})}{D(0, \frac{2}{n})} f(\frac{1}{n}, y_1) + 0 \right] + \frac{c}{a(0)B_1(b)}.$$

By (4.7) we also have

$$(4.14) \quad \frac{-y_0}{D(0, \frac{1}{n})} + \frac{D(0, \frac{2}{n})}{D(0, \frac{1}{n})D(\frac{1}{n}, \frac{2}{n})} y_1 + \frac{-y_2}{D(\frac{1}{n}, \frac{2}{n})} = hf(\frac{1}{n}, y_1) + \frac{h^2}{12} \left[\frac{-f(0, y_0)}{p(0)D(0, \frac{1}{n})} + \frac{D(0, \frac{2}{n})f(\frac{1}{n}, y_1)}{p(\frac{1}{n})D(0, \frac{1}{n})D(\frac{1}{n}, \frac{2}{n})} + \frac{-f(\frac{2}{n}, y_2)}{p(\frac{2}{n})D(\frac{1}{n}, \frac{2}{n})} \right].$$

If we eliminate y_2 in (4.13) by (4.14) we get

$$(4.15) \quad \frac{a(\frac{1}{n})}{a(0)D(0,\frac{1}{n})} y_0 - \frac{y_1}{D(0,\frac{1}{n})} = \frac{h}{3} \left[f(0, y_0) + \frac{D(\frac{1}{n}, \frac{2}{n})}{D(0, \frac{2}{n})} f(\frac{1}{n}, y_1) \right] -$$

$$- \frac{h^2}{12} \frac{D(\frac{1}{n}, \frac{2}{n})}{D(0, \frac{2}{n})} \left[\frac{-f(0, y_0)}{p(0)D(0, \frac{1}{n})} + \frac{D(0, \frac{2}{n})f(\frac{1}{n}, y_1)}{p(\frac{1}{n})D(0, \frac{1}{n})D(\frac{1}{n}, \frac{2}{n})} + \frac{-f(\frac{2}{n}, y_2)}{p(\frac{2}{n})D(\frac{1}{n}, \frac{2}{n})} \right] +$$

$$+ \frac{c}{a(0)B_1(b)}.$$

If $b_{22} > 0$ we get by the same method

$$(4.16) \quad \frac{-y_{n-1}}{D(\frac{n-1}{n}, 1)} + \frac{b(\frac{n-1}{n})}{b(1)D(\frac{n-1}{n}, 1)} y_n = \frac{h}{3} \left[\frac{D(\frac{n-2}{n}, \frac{n-1}{n})}{D(\frac{n-2}{n}, 1)} f(\frac{n-1}{n}, y_{n-1}) + \right.$$

$$+ f(1, y_n) \left. \right] - \frac{h^2}{12} \frac{D(\frac{n-2}{n}, \frac{n-1}{n})}{D(\frac{n-2}{n}, 1)} \left[- \frac{f(\frac{n-2}{n}, y_{n-2})}{p(\frac{n-2}{n})D(\frac{n-2}{n}, \frac{n-1}{n})} + \right.$$

$$+ \frac{D(\frac{n-2}{n}, 1)f(\frac{n-1}{n}, y_{n-1})}{p(\frac{n-1}{n})D(\frac{n-2}{n}, \frac{n-1}{n})D(\frac{n-1}{n}, 1)} - \frac{f(1, y_n)}{p(1)D(\frac{n-1}{n}, 1)} \left. \right] + \frac{d}{b(1)B_2(a)}.$$

Equations (4.7), (4.15), and (4.16) define an $O(h^4)$ -approximation of the determinant formulation of the boundary value problem (2.1).

We will now obtain that approximation of the integral equation which corresponds to (4.7), (4.15), and (4.16). We write the integral equation (2.5) in the following form

$$\begin{aligned}
 (4.17) \quad y_{\frac{i}{n}} &= b\left(\frac{i}{n}\right)a(0) \int_0^{\frac{2}{n}} \frac{D(t, \frac{2}{n})}{D(0, \frac{2}{n})} f \, dt + a\left(\frac{i}{n}\right)b(1) \int_{\frac{n-2}{n}}^1 \frac{D(\frac{n-2}{n}, t)}{D(\frac{n-2}{n}, 1)} f \, dt + \\
 &+ \left\{ \int_0^1 G\left(\frac{i}{n}, t\right) f \, dt - b\left(\frac{i}{n}\right)a(0) \int_0^{\frac{2}{n}} \frac{D(t, \frac{2}{n})}{D(0, \frac{2}{n})} f \, dt - \right. \\
 &\left. - a\left(\frac{i}{n}\right)b(1) \int_{\frac{n-2}{n}}^1 \frac{D(\frac{n-2}{n}, t)}{D(\frac{n-2}{n}, 1)} f \, dt \right\} + \frac{b(\frac{i}{n})c}{B_1(b)} + \frac{a(\frac{i}{n})d}{B_2(a)}, \quad i = 0, \dots, n,
 \end{aligned}$$

which make sense if $D(0, \frac{2}{n})D(\frac{n-2}{n}, 1) \neq 0$. If we approximate the first two integrals by Simpson's rule and the other integrals by the trapezoidal rule with "end correction", we get

$$\begin{aligned}
 (4.18) \quad y_i &= b\left(\frac{i}{n}\right)a(0) \frac{h}{3} \left[f(0, y_0) + \frac{D(\frac{1}{n}, \frac{2}{n})}{D(0, \frac{2}{n})} f\left(\frac{1}{n}, y_1\right) \right] + \\
 &+ \sum_{j=1}^{n-1} hG\left(\frac{i}{n}, \frac{j}{n}\right) f\left(\frac{j}{n}, y_j\right) + a\left(\frac{i}{n}\right)b(1) \frac{h}{3} \left[\frac{D(\frac{n-2}{n}, \frac{n-1}{n})}{D(\frac{n-2}{n}, 1)} f\left(\frac{n-1}{n}, y_{n-1}\right) + \right. \\
 &+ \left. f(1, y_n) \right] + \frac{h^2}{12} \left[- \frac{b(\frac{i}{n})a(\frac{2}{n})}{D(0, \frac{2}{n})} \frac{f(0, y_0)}{p(0)} + \frac{b(\frac{i}{n})a(0)}{D(0, \frac{2}{n})} \frac{f(\frac{2}{n}, y_2)}{p(\frac{2}{n})} + \right. \\
 &+ \left. \frac{f(\frac{1}{n}, y_1)}{p(\frac{1}{n})} + \frac{a(\frac{i}{n})b(1)}{D(\frac{n-2}{n}, 1)} \frac{f(\frac{n-2}{n}, y_{n-2})}{p(\frac{n-2}{n})} - \frac{a(\frac{i}{n})b(\frac{n-2}{n})}{D(\frac{n-2}{n}, 1)} \frac{f(1, y_n)}{p(1)} \right] + \\
 &+ \frac{b(\frac{i}{n})c}{B_1(b)} + \frac{a(\frac{i}{n})d}{B_2(a)},
 \end{aligned}$$

for $i = 0, \dots, n$.

Now consider the discrete determinant formulation defined by (4.7), (4.15), and (4.16). The left hand side of this system defines a

matrix T . Then $T^{-1} = G = (g_{i,j})_{i,j=0}^n$, where

$$g_{i,j} = \begin{cases} b(\frac{i}{n})a(\frac{j}{n}), & i \geq j, \\ a(\frac{i}{n})b(\frac{j}{n}), & i \leq j. \end{cases}$$

If we multiply both sides of the system (4.7), (4.15), (4.16) by G , we get (after "some" computation) (4.18). Thus we see that the combined Simpson's rule and the trapezoidal rule with "end correction" commutes with the determinant formulation of the problem.

Another method to approximate the integrals in the determinant formulation given by (3.5), (3.7), and (3.9) is to replace f by its Lagrange interpolants. Consider the grid $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$ and the integrals

$$(4.19) \quad \int_0^{x_1} \Delta_0(t) f(t, y(t)) dt,$$

$$(4.20) \quad \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) f(t, y(t)) dt, \quad i = 1, \dots, n-1,$$

$$(4.21) \quad \int_{x_{n-1}}^1 \Delta_n(t) f(t, y(t)) dt.$$

For each of these integrals consider the Lagrange interpolant $f_i(t)$ of f at some set of points $\{x_{i-r_i}, \dots, x_i, \dots, x_{i+s_i}\}$ for $i = 0, 1, \dots, n$, where $r_i, s_i \geq 0$ are integers. If the integrals (4.19), (4.20), (4.21) can be evaluated exactly when f is replaced by a polynomial we get the following product integration approximation of the determinant formulation of (2.1)

$$(4.22) \quad \frac{a(x_1)}{a(0)D(0, x_1)} y_0 - \frac{y_1}{D(0, x_1)} = \int_0^{x_1} \Delta_0(t) f_0(t) dt + \frac{c}{a(0)B_1(b)},$$

if $a_{12} > 0$,

$$(4.23) \quad \frac{-y_{i-1}}{D(x_{i-1}, x_i)} + \frac{D(x_{i-1}, x_{i+1})}{D(x_{i-1}, x_i)D(x_i, x_{i+1})} y_i + \frac{-y_{i+1}}{D(x_i, x_{i+1})} =$$

$$= \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) f_i(t) dt, \quad i = 1, \dots, n-1,$$

and

$$(4.24) \quad \frac{-y_{n-1}}{D(x_{n-1}, 1)} + \frac{b(x_{n-1})}{b(1)D(x_{n-1}, 1)} y_n = \int_{x_{n-1}}^1 \Delta_n(t) f_n(t) dt + \frac{d}{b(1)B_2(a)},$$

if $b_{22} > 0$, where $f_i(t) = f_i(t; y_{i-r_i}, \dots, y_i, \dots, y_{i+s_i})$, $y_i \approx y(x_i)$.

The left hand side of (4.22), (4.23), and (4.24) defines a

matrix T and $T^{-1} = G = (g_{i,j})$, where

$$g_{i,j} = \begin{cases} b(x_i)a(x_j), & i \geq j, \\ a(x_i)b(x_j), & i \leq j. \end{cases}$$

If we multiply these equations by G and order the sums of the

right hand side after the integration over the intervals $[x_{j-1}, x_j]$,

we get

$$\begin{aligned}
 (4.25) \quad \left\{ \begin{aligned}
 y_0 &= a(0) \sum_{j=1}^n \int_{x_{j-1}}^{x_j} \left[b(x_{j-1}) \Delta_{j-1}(t) f_{j-1}(t) + b(x_j) \Delta_j(t) f_j(t) \right] dt + \\
 &+ \frac{cb(0)}{B_1(b)} + \frac{da(0)}{B_2(a)}, \\
 y_i &= b(x_i) \sum_{j=1}^i \int_{x_{j-1}}^{x_j} \left[a(x_{j-1}) \Delta_{j-1}(t) f_{j-1}(t) + a(x_j) \Delta_j(t) f_j(t) \right] dt + \\
 &+ a(x_i) \sum_{j=i+1}^n \int_{x_{j-1}}^{x_j} \left[b(x_{j-1}) \Delta_{j-1}(t) f_{j-1}(t) + b(x_j) \Delta_j(t) f_j(t) \right] dt + \\
 &+ \frac{cb(x_i)}{B_1(b)} + \frac{da(x_i)}{B_2(a)}, \quad i = 1, \dots, n-1, \\
 y_n &= b(1) \sum_{j=1}^n \int_{x_{j-1}}^{x_j} \left[a(x_{j-1}) \Delta_{j-1}(t) f_{j-1}(t) + a(x_j) \Delta_j(t) f_j(t) \right] dt + \\
 &+ \frac{cb(1)}{B_1(b)} + \frac{da(1)}{B_2(a)}.
 \end{aligned} \right.
 \end{aligned}$$

Put

$$(4.26) \quad H_a(t, f_{j-1}, f_j) = a(x_{j-1}) \Delta_{j-1}(t) f_{j-1}(t) + a(x_j) \Delta_j(t) f_j(t),$$

and

$$(4.27) \quad H_b(t, f_{j-1}, f_j) = b(x_{j-1}) \Delta_{j-1}(t) f_{j-1}(t) + b(x_j) \Delta_j(t) f_j(t),$$

for $x_{j-1} \leq t \leq x_j$, $j = 1, \dots, n$. Then we have

$$(4.28) \quad H_a(t, f_{j-1}, f_j) = \frac{1}{D(x_{j-1}, x_j)} \begin{vmatrix} b(x_{j-1}) & a(x_{j-1}) & a(x_{j-1})f_{j-1}(t) \\ b(t) & a(t) & 0 \\ b(x_j) & a(x_j) & a(x_j)f_j(t) \end{vmatrix}$$

$$(4.29) \quad H_b(t, f_{j-1}, f_j) = \frac{1}{D(x_{j-1}, x_j)} \begin{vmatrix} b(x_{j-1}) & a(x_{j-1}) & b(x_{j-1})f_{j-1}(t) \\ b(t) & a(t) & 0 \\ b(x_j) & a(x_j) & b(x_j)f_j(t) \end{vmatrix}$$

for $x_{j-1} \leq t \leq x_j$, $j = 1, \dots, n$. Now (4.25) can be written as

$$(4.30) \quad y_i = b(x_i) \sum_{j=1}^i \int_{x_{j-1}}^{x_j} H_a(t, f_{j-1}, f_j) dt + \\ + a(x_i) \sum_{j=i+1}^n \int_{x_{j-1}}^{x_j} H_b(t, f_{j-1}, f_j) dt + \frac{cb(x_i)}{B_1(b)} + \frac{da(x_i)}{B_2(a)},$$

for $i = 0, 1, \dots, n$, where $\sum_{j=1}^0 = \sum_{j=n+1}^n = 0$. We also note that the integral equation (2.5) can be written as

$$(4.31) \quad y(x_i) = b(x_i) \sum_{j=1}^i \int_{x_{j-1}}^{x_j} H_a(t, f, f) dt + \\ + a(x_i) \sum_{j=i+1}^n \int_{x_{j-1}}^{x_j} H_b(t, f, f) dt + \frac{cb(x_i)}{B_1(b)} + \frac{da(x_i)}{B_2(a)},$$

for $i = 0, 1, \dots, n$.

If the integrals in (4.22), (4.23), and (4.24) cannot be evaluated exactly then we can approximate these integrals by any quadrature rule. We then get product quadrature formulas for these integrals (see Boland and Duris [8]). To give an example consider an integral in (4.23) where we assume that $f_i(t)$ is the Lagrange

interpolant of f at the points x_{i-1}, x_i, x_{i+1} ,

$$f_i(t) = p_{i-1}(t)f(x_{i-1}, y_{i-1}) + p_i(t)f(x_i, y_i) + p_{i+1}(t)f(x_{i+1}, y_{i+1}), \quad x_{i-1} \leq t \leq x_{i+1}.$$

If we split the integral over the intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$

and make use of a Gauss rule with three abscissas we get

$$\begin{aligned} (4.32) \quad \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) f_i(t) dt &= \int_{x_{i-1}}^{x_i} \Delta_i(t) f_i(t) dt + \int_{x_i}^{x_{i+1}} \Delta_i(t) f_i(t) dt \\ &\approx w_1 \Delta_i(\tau_1) [p_{i-1}(\tau_1) f(x_{i-1}, y_{i-1}) + p_i(\tau_1) f(x_i, y_i) + p_{i+1}(\tau_1) f(x_{i+1}, y_{i+1})] + \\ &+ w_2 \Delta_i(\tau_2) [p_{i-1}(\tau_2) f(x_{i-1}, y_{i-1}) + p_i(\tau_2) f(x_i, y_i) + p_{i+1}(\tau_2) f(x_{i+1}, y_{i+1})] + \dots + \\ &+ w_6 \Delta_i(\tau_6) [p_{i-1}(\tau_6) f(x_{i-1}, y_{i-1}) + p_i(\tau_6) f(x_i, y_i) + p_{i+1}(\tau_6) f(x_{i+1}, y_{i+1})], \end{aligned}$$

where $\tau_1, \tau_2, \tau_3 \in [x_{i-1}, x_i]$, $\tau_4, \tau_5, \tau_6 \in [x_i, x_{i+1}]$. In vector form

(4.32) can be written as

$$\begin{aligned} (4.33) \quad \int_{x_{i-1}}^{x_{i+1}} \Delta_i(t) f_i(t) dt &\approx \\ &\approx (\Delta_i(\tau_1) \dots \Delta_i(\tau_6)) \begin{bmatrix} w_1 p_{i-1}(\tau_1) & w_1 p_i(\tau_1) & w_1 p_{i+1}(\tau_1) \\ \vdots & \vdots & \vdots \\ w_6 p_{i-1}(\tau_6) & w_6 p_i(\tau_6) & w_6 p_{i+1}(\tau_6) \end{bmatrix} \begin{bmatrix} f(x_{i-1}, y_{i-1}) \\ f(x_i, y_i) \\ f(x_{i+1}, y_{i+1}) \end{bmatrix}. \end{aligned}$$

Another method is to approximate f with spline functions.

Consider the boundary value problem (2.8). The integral equation formulation of this problem is given by (2.13). The determinant formulation of the integral equation (2.13), on the grid $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$, is defined by (3.5), (3.7), and (3.9). For $0 \leq t_1 < t_2 \leq 1$ we have

$$\begin{aligned}
 D(t_1, t_2) &= \begin{vmatrix} b(t_1) & a(t_1) \\ b(t_2) & a(t_2) \end{vmatrix} = \\
 &= \frac{1}{D_p} \begin{vmatrix} b_{21}p(1)[P(t_1) - P(1)] - b_{22} & a_{11}p(0)P(t_1) + a_{12} \\ b_{21}p(1)[P(t_2) - P(1)] - b_{22} & a_{11}p(0)P(t_2) + a_{12} \end{vmatrix} = \\
 &= \frac{P(t_2) - P(t_1)}{D_p} \begin{vmatrix} b_{21}p(1)[P(t_1) - P(1)] - b_{22} & a_{11}p(0)P(t_1) + a_{12} \\ b_{21}p(1) & a_{11}p(0) \end{vmatrix} = \\
 &= - [P(t_2) - P(t_1)] = - \int_{t_1}^{t_2} dt/p(t).
 \end{aligned}$$

Hence for the boundary value problem (2.8) we have

$$(4.34) \quad \begin{cases} D(x_{i-1}, x_i) = - \int_{x_{i-1}}^{x_i} dt/p(t), & i = 1, \dots, n, \\ D(x_{i-1}, x_{i+1}) = D(x_{i-1}, x_i) + D(x_i, x_{i+1}), & i = 1, \dots, n-1. \end{cases}$$

Since $D(x_{i-1}, x_i) < 0$, $i = 1, \dots, n$, the determinant formulation, given by (3.5), (3.7), and (3.9), is always equivalent to the integral equation (2.13) on the given grid. If we approximate the integrals of the right hand sides of (3.5), (3.7), and (3.9) by the trapezoidal rule then by (4.1) we get $(y_i \approx y(x_i))$

$$(4.35) \left\{ \begin{array}{l} \left[\frac{1}{D(0, x_1)} - \frac{a_{11}p(0)}{a_{12}} \right] y_0 - \frac{y_1}{D(0, x_1)} = w_0 f(0, y_0) - \frac{cp(0)}{a_{12}}, \\ \text{if } a_{12} > 0, \\ \\ \frac{-y_{i-1}}{D(x_{i-1}, x_i)} + \left[\frac{1}{D(x_{i-1}, x_i)} + \frac{1}{D(x_i, x_{i+1})} \right] y_i + \frac{-y_{i+1}}{D(x_i, x_{i+1})} = \\ = w_i f(x_i, y_i), \quad i = 1, \dots, n-1, \\ \\ \frac{-y_{n-1}}{D(x_{n-1}, 1)} + \left[\frac{1}{D(x_{n-1}, 1)} - \frac{b_{21}p(1)}{b_{22}} \right] y_n = w_n f(1, y_n) - \frac{dp(1)}{b_{22}}, \\ \text{if } b_{22} > 0. \end{array} \right.$$

In general the integral

$$\int_{x_{i-1}}^{x_i} dt/p(t)$$

cannot be evaluated exactly, but by a quadrature formula this integral can be approximated to any degree of precision. We will consider the midpoint rule

$$\int_{x_1}^{x_2} F(x) dx = (x_2 - x_1) F\left(\frac{x_1 + x_2}{2}\right) + \frac{(x_2 - x_1)^3}{4} F^{(2)}\left(\frac{\xi}{2}\right), \quad x_1 < \xi < x_2,$$

for $F \in C^2$. Hence if $p \in C^2[0, 1]$ we have

$$D(x_{i-1}, x_i) \approx - \frac{x_i - x_{i-1}}{p\left(\frac{x_{i-1} + x_i}{2}\right)}, \quad i = 1, \dots, n,$$

and (4.35) takes the form

$$\begin{aligned}
 & - \left[\frac{p(\frac{x_1}{2})}{x_1} + \frac{a_{11}p(0)}{a_{12}} \right] y_0 + \frac{p(\frac{x_1}{2})}{x_1} y_1 = \frac{x_1}{2} f(0, y_0) - \frac{cp(0)}{a_{12}}, \\
 & \text{if } a_{12} > 0, \\
 & \frac{p(\frac{x_{i-1}+x_i}{2})}{x_i - x_{i-1}} y_{i-1} - \left[\frac{p(\frac{x_{i-1}+x_i}{2})}{x_i - x_{i-1}} + \frac{p(\frac{x_i+x_{i+1}}{2})}{x_{i+1} - x_i} \right] y_i + \\
 & + \frac{p(\frac{x_i+x_{i+1}}{2})}{x_{i+1} - x_i} y_{i+1} = \frac{x_{i+1} - x_{i-1}}{2} f(x_i, y_i), \quad i = 1, \dots, n-1, \\
 & \frac{p(\frac{x_{n-1}+1}{2})}{1 - x_{n-1}} y_{n-1} - \left[\frac{p(\frac{x_{n-1}+1}{2})}{1 - x_{n-1}} + \frac{b_{21}p(1)}{b_{22}} \right] y_n = \frac{1 - x_{n-1}}{2} f(1, y_n) - \\
 & - \frac{dp(1)}{b_{22}}, \quad \text{if } b_{22} > 0.
 \end{aligned}
 \tag{4.36}$$

Now let $x_i = i/n$, $i = 0, \dots, n$, $h = 1/n$, and put $p_{i+\frac{1}{2}} = p(\frac{i+\frac{1}{2}}{n})$.

Then (4.36) can be written as

$$\begin{aligned}
 & - \left[p_{\frac{1}{2}} + h \frac{a_{11}p_0}{a_{12}} \right] y_0 + p_{\frac{1}{2}} y_1 = \frac{h^2}{2} f(0, y_0) - h \frac{cp_0}{a_{12}}, \quad \text{if } a_{12} > 0, \\
 & p_{i+\frac{1}{2}}(y_{i+1} - y_i) - p_{i-\frac{1}{2}}(y_i - y_{i-1}) = h^2 f(\frac{i}{n}, y_i), \quad i = 1, \dots, n-1, \\
 & p_{n-\frac{1}{2}} y_{n-1} - \left[p_{n-\frac{1}{2}} + h \frac{b_{21}p_n}{b_{22}} \right] y_n = \frac{h^2}{2} f(1, y_n) - h \frac{dp_n}{b_{22}}, \quad \text{if } b_{22} > 0.
 \end{aligned}
 \tag{4.37}$$

We recognize (4.37) as the finite difference approximation of the boundary value problem (2.8). However, we say that (4.37) is the approximation of the determinant formulation of (2.8) by the trapezoidal rule and the midpoint rule.

We will now give an $O(h^4)$ -approximation for the boundary value problem (2.8). Let $x_i = \frac{i}{n}$, $i = 0, \dots, n$, $h = 1/n$. By (4.7), (4.15), and (4.16) we get

$$\begin{aligned}
 (4.38) \quad & \left[\frac{1}{D(0, \frac{1}{n})} - \frac{a_{11}p(0)}{a_{12}} \right] y_0 - \frac{y_1}{D(0, \frac{1}{n})} = \frac{h}{3} \left[f(0, y_0) + \frac{D(\frac{1}{n}, \frac{2}{n})}{D(0, \frac{2}{n})} f(\frac{1}{n}, y_1) \right] - \\
 & - \frac{h^2}{12} \frac{D(\frac{1}{n}, \frac{2}{n})}{D(0, \frac{2}{n})} \left\{ \left[\frac{f(\frac{1}{n}, y_1)}{p(\frac{1}{n})} - \frac{f(0, y_0)}{p(0)} \right] \frac{1}{D(0, \frac{1}{n})} - \right. \\
 & \left. - \left[\frac{f(\frac{2}{n}, y_2)}{p(\frac{2}{n})} - \frac{f(\frac{1}{n}, y_1)}{p(\frac{1}{n})} \right] \frac{1}{D(\frac{1}{n}, \frac{2}{n})} \right\} - \frac{cp(0)}{a_{12}}, \quad \text{if } a_{12} > 0,
 \end{aligned}$$

$$\begin{aligned}
 (4.39) \quad & \frac{-y_{i-1}}{D(\frac{i-1}{n}, \frac{i}{n})} + \left[\frac{1}{D(\frac{i-1}{n}, \frac{i}{n})} + \frac{1}{D(\frac{i}{n}, \frac{i+1}{n})} \right] y_i + \frac{-y_{i+1}}{D(\frac{i}{n}, \frac{i+1}{n})} = \\
 & = hf(\frac{i}{n}, y_i) + \frac{h^2}{12} \left\{ \frac{1}{D(\frac{i-1}{n}, \frac{i}{n})} \left[\frac{f(\frac{i}{n}, y_i)}{p(\frac{i}{n})} - \frac{f(\frac{i-1}{n}, y_{i-1})}{p(\frac{i-1}{n})} \right] - \right. \\
 & \left. - \frac{1}{D(\frac{i}{n}, \frac{i+1}{n})} \left[\frac{f(\frac{i+1}{n}, y_{i+1})}{p(\frac{i+1}{n})} - \frac{f(\frac{i}{n}, y_i)}{p(\frac{i}{n})} \right] \right\}, \quad i = 1, \dots, n-1,
 \end{aligned}$$

and

$$\begin{aligned}
 (4.40) \quad & \frac{-y_{n-1}}{D(\frac{n-1}{n}, 1)} + \left[\frac{1}{D(\frac{n-1}{n}, 1)} - \frac{b_{21}p(1)}{b_{22}} \right] y_n = \\
 & = \frac{h}{3} \left[\frac{D(\frac{n-2}{n}, \frac{n-1}{n})}{D(\frac{n-2}{n}, 1)} f(\frac{n-1}{n}, y_{n-1}) + f(1, y_n) \right] - \\
 & - \frac{h^2}{12} \frac{D(\frac{n-2}{n}, \frac{n-1}{n})}{D(\frac{n-2}{n}, 1)} \left\{ \frac{1}{D(\frac{n-2}{n}, \frac{n-1}{n})} \left[\frac{f(\frac{n-1}{n}, y_{n-1})}{p(\frac{n-1}{n})} - \frac{f(\frac{n-2}{n}, y_{n-2})}{p(\frac{n-2}{n})} \right] - \right. \\
 & \left. - \frac{1}{D(\frac{n-1}{n}, 1)} \left[\frac{f(1, y_n)}{p(1)} - \frac{f(\frac{n-1}{n}, y_{n-1})}{p(\frac{n-1}{n})} \right] \right\} - \frac{dp(1)}{b_{22}}, \quad \text{if } b_{22} > 0,
 \end{aligned}$$

where

$$(4.41) \quad \begin{cases} D(\frac{i-1}{n}, \frac{i}{n}) = - \int_{\frac{i-1}{n}}^{\frac{i}{n}} dt/p(t), & i = 1, \dots, n, \\ D(0, \frac{2}{n}) = D(0, \frac{1}{n}) + D(\frac{1}{n}, \frac{2}{n}), & D(\frac{n-2}{n}, 1) = D(\frac{n-2}{n}, \frac{n-1}{n}) + D(\frac{n-1}{n}, 1). \end{cases}$$

If the integral $\int_{\frac{i-1}{n}}^{\frac{i}{n}} dt/p(t)$ cannot be evaluated exactly we can use

Simpson's rule

$$\int_{\frac{i-1}{n}}^{\frac{i}{n}} dt/p(t) \approx \frac{h}{6} \left[\frac{1}{p(\frac{i-1}{n})} + \frac{4}{p(\frac{i-\frac{1}{2}}{n})} + \frac{1}{p(\frac{i}{n})} \right],$$

for $i = 1, \dots, n$.

For $p(t) \equiv 1$, $x_i = i/n$, $i = 0, \dots, n$, $h = 1/n$, (4.36) takes the form

$$(4.42) \quad \begin{cases} - \left[1 + h \frac{a_{11}}{a_{12}} \right] y_0 + y_1 = \frac{h^2}{2} f(0, y_0) - h \frac{c}{a_{12}}, & \text{if } a_{12} > 0, \\ y_{i-1} - 2y_i + y_{i+1} = h^2 f\left(\frac{i}{n}, y_i\right), & i = 1, \dots, n-1, \\ y_{n-1} - \left[1 + h \frac{b_{21}}{b_{22}} \right] y_n = \frac{h^2}{2} f(1, y_n) - h \frac{d}{b_{22}}, & \text{if } b_{22} > 0, \end{cases}$$

and (4.38), (4.39), and (4.40) take the form

$$(4.43) \quad \begin{cases} - \left[1 + h \frac{a_{11}}{a_{12}} \right] y_0 + y_1 = \frac{h^2}{3} \left[f(0, y_0) + \frac{1}{2} f\left(\frac{1}{n}, y_1\right) \right] - \\ - \frac{h^2}{12} \frac{1}{2} \left\{ \left[f\left(\frac{2}{n}, y_2\right) - f\left(\frac{1}{n}, y_1\right) \right] - \left[f\left(\frac{1}{n}, y_1\right) - f(0, y_0) \right] \right\} - \\ - h \frac{c}{a_{12}}, & \text{if } a_{12} > 0, \\ y_{i-1} - 2y_i + y_{i+1} = h^2 f\left(\frac{i}{n}, y_i\right) + \frac{h^2}{12} \left\{ \left[f\left(\frac{i+1}{n}, y_{i+1}\right) - f\left(\frac{i}{n}, y_i\right) \right] - \right. \\ \left. - \left[f\left(\frac{i}{n}, y_i\right) - f\left(\frac{i-1}{n}, y_{i-1}\right) \right] \right\}, & i = 1, \dots, n-1, \\ y_{n-1} - \left[1 + h \frac{b_{21}}{b_{22}} \right] y_n = \frac{h^2}{3} \left[\frac{1}{2} f\left(\frac{n-1}{n}, y_{n-1}\right) + f(1, y_n) \right] - \\ - \frac{h^2}{12} \frac{1}{2} \left\{ \left[f(1, y_n) - f\left(\frac{n-1}{n}, y_{n-1}\right) \right] - \left[f\left(\frac{n-1}{n}, y_{n-1}\right) - f\left(\frac{n-2}{n}, y_{n-2}\right) \right] \right\} - \\ - h \frac{d}{b_{22}}, & \text{if } b_{22} > 0. \end{cases}$$

Remark.

In all the discrete equations above the right hand sides have been written according to Pohl [18].

We note that there is never any reason to use points outside the interval $[0,1]$ to approximate boundary value problems at the end-points when a_{12} or $b_{22} > 0$. For the finite difference approximation of the boundary value problem

$$\begin{cases} y'' = f(t), & 0 < t < 1, \\ a_{11}y(0) - a_{12}y'(0) = c, & B_2y = d, \end{cases}$$

the following trick is frequently used. Consider the finite difference approximation of the differential equation and the first boundary condition given by

$$\begin{cases} y_{-1} - 2y_0 + y_1 = h^2 f_0, \\ a_{11}y_0 - a_{12}(y_1 - y_{-1})/(2h) = c. \end{cases}$$

If we eliminate y_{-1} we get

$$-2 \left[1 - h \frac{a_{11}}{a_{12}} \right] y_0 + 2y_1 = h^2 f_0 - \frac{2hc}{a_{12}},$$

which is the same equation as the first equation in (4.42).

We will now consider the boundary value problem (2.15). The determinant formulation of the integral equation (2.26) is defined by (3.5), (3.7), and (3.9) on the grid $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$.

For $0 \leq t_1 < t_2 \leq 1$ we have

$$(4.44) \quad D(t_1, t_2) = \frac{\sinh(k(t_1 - t_2))}{k} < 0, \quad k > 0.$$

Let $x_i = \frac{i}{n}$, $i = 0, \dots, n$, $h = 1/n$. Then we have

$$(4.45) \quad D\left(\frac{i-1}{n}, \frac{i}{n}\right) = - \frac{\sinh\left(\frac{k}{n}\right)}{k} < 0$$

for $i = 1, \dots, n$. Hence the determinant formulation of the boundary value problem (2.15) is always equivalent to the integral equation formulation on the given grid. If we approximate the determinant formulation of (2.15) by the trapezoidal rule, then by (4.1) and (2.18) we get

$$(4.46) \left\{ \begin{array}{l} - \left[\cosh\left(\frac{k}{n}\right) + \frac{a_{11} \sinh\left(\frac{k}{n}\right)}{ka_{12}} \right] y_0 + y_1 = \frac{\sinh\left(\frac{k}{n}\right)}{2kn} f(0, y_0) - \frac{c \sinh\left(\frac{k}{n}\right)}{ka_{12}}, \\ \text{if } a_{12} > 0, \\ y_{i-1} - 2\cosh\left(\frac{k}{n}\right)y_i + y_{i+1} = \frac{\sinh\left(\frac{k}{n}\right)}{kn} f\left(\frac{i}{n}, y_i\right), \quad i = 1, \dots, n-1, \\ y_{n-1} - \left[\cosh\left(\frac{k}{n}\right) + \frac{b_{21} \sinh\left(\frac{k}{n}\right)}{kb_{22}} \right] y_n = \frac{\sinh\left(\frac{k}{n}\right)}{2kn} f(1, y_n) - \frac{d \sinh\left(\frac{k}{n}\right)}{kb_{22}}, \\ \text{if } b_{22} > 0. \end{array} \right.$$

By (4.7), (4.15), (4.16) we get an $O(h^4)$ -approximation of the boundary value problem (2.15) by

$$(4.47) \left\{ \begin{array}{l} - \left[\cosh\left(\frac{k}{n}\right) + \frac{a_{11} \sinh\left(\frac{k}{n}\right)}{ka_{12}} \right] y_0 + y_1 = \\ = \frac{\sinh\left(\frac{k}{n}\right)}{3kn} \left[f(0, y_0) + \frac{1}{2\cosh\left(\frac{k}{n}\right)} f\left(\frac{1}{n}, y_1\right) \right] - \\ - \frac{h^2}{12} \frac{1}{2\cosh\left(\frac{k}{n}\right)} \left\{ \left[\left(f\left(\frac{2}{n}, y_2\right) - f\left(\frac{1}{n}, y_1\right) \right) - \left(f\left(\frac{1}{n}, y_1\right) - f(0, y_0) \right) \right] - \right. \\ \left. - 4\sinh^2\left(\frac{k}{2n}\right) f\left(\frac{1}{n}, y_1\right) \right\} - \frac{c \sinh\left(\frac{k}{n}\right)}{ka_{12}}, \quad \text{if } a_{12} > 0, \\ y_{i-1} - 2\cosh\left(\frac{k}{n}\right)y_i + y_{i+1} = \frac{\sinh\left(\frac{k}{n}\right)}{kn} f\left(\frac{i}{n}, y_i\right) + \\ + \frac{h^2}{12} \left\{ \left[\left(f\left(\frac{i+1}{n}, y_{i+1}\right) - f\left(\frac{i}{n}, y_i\right) \right) - \left(f\left(\frac{i}{n}, y_i\right) - f\left(\frac{i-1}{n}, y_{i-1}\right) \right) \right] - \right. \\ \left. - 4\sinh^2\left(\frac{k}{2n}\right) f\left(\frac{i}{n}, y_i\right) \right\}, \quad i = 1, \dots, n-1, \\ y_{n-1} - \left[\cosh\left(\frac{k}{n}\right) + \frac{b_{21} \sinh\left(\frac{k}{n}\right)}{kb_{22}} \right] y_n = \\ = \frac{\sinh\left(\frac{k}{n}\right)}{3kn} \left[\frac{1}{2\cosh\left(\frac{k}{n}\right)} f\left(\frac{n-1}{n}, y_{n-1}\right) + f(1, y_n) \right] - \\ - \frac{h^2}{12} \frac{1}{2\cosh\left(\frac{k}{n}\right)} \left\{ \left[\left(f(1, y_n) - f\left(\frac{n-1}{n}, y_{n-1}\right) \right) - \left(f\left(\frac{n-1}{n}, y_{n-1}\right) - \right. \right. \right. \\ \left. \left. - f\left(\frac{n-2}{n}, y_{n-2}\right) \right) \right] - 4\sinh^2\left(\frac{k}{2n}\right) f\left(\frac{n-1}{n}, y_{n-1}\right) \right\} - \frac{d \sinh\left(\frac{k}{n}\right)}{kb_{22}}, \quad \text{if } b_{22} > 0. \end{array} \right.$$

When Newton's method is used to solve (4.46) or (4.47) we write

$$\begin{aligned} y_{i-1} - 2\cosh\left(\frac{k}{n}\right)y_i + y_{i+1} &= \\ &= \left[(y_{i+1} - y_i) - (y_i - y_{i-1})\right] - 4\sinh^2\left(\frac{k}{2n}\right)y_i \end{aligned}$$

to avoid round-off errors (see Pohl [18]).

An $O(h^2)$ discretization of the boundary value problem (2.23) is given by (4.46) if we replace "cosh" by "cos" and "sinh" by "sin". We get an $O(h^4)$ discretization of (2.23) if we replace "cosh" by "cos", "sinh" by "sin", and " $\sinh^2$ " by " $-\sin^2$ " in (4.47). However we must have $D(\frac{i-1}{n}, \frac{i}{n}) = -\sin(\frac{k}{n}) \neq 0$ which is fulfilled if n is sufficiently large.

For the linear boundary value problem

$$(4.48) \quad \begin{cases} y'' - k^2 y = f(t), & 0 < t < 1, \\ y(0) = c, & y(1) = d, \end{cases}$$

the determinant formulation is, by (3.11),

$$(4.49) \quad y\left(\frac{i-1}{n}\right) - \frac{D(2h)}{D(h)} y\left(\frac{i}{n}\right) + y\left(\frac{i+1}{n}\right) = h \int_{-1}^1 D(h(1 - |t|)) f\left(\frac{i+t}{n}\right) dt$$

for $i = 1, \dots, n-1$, $h = 1/n$, where

$$(4.50) \quad D(h(1 - |t|)) = \begin{cases} h(1 - |t|), & k = 0, \\ \frac{1}{k} \sinh\left(\frac{k}{n}(1 - |t|)\right), & k > 0, \\ \frac{1}{k} \sin\left(\frac{k}{n}(1 - |t|)\right), & k := \sqrt{-1}k, \quad k > 0. \end{cases}$$

We assume that k^2 is not an eigenvalue of the homogeneous problem.

Hence we can use the same Gauss quadrature formula, with weight function $D(h(1 - |t|))$, to approximate the integrals in (4.49).

5. DISCRETIZATION ERROR

We will consider the boundary value problem (2.1), which we repeat here for convenience,

$$(5.1) \quad \begin{cases} Ly = (py')' + qy = f(t, y(t)), & 0 < t < 1, \\ B_1 y = a_{11}y(0) - a_{12}y'(0) = c, \\ B_2 y = b_{21}y(1) + b_{22}y'(1) = d. \end{cases}$$

When zero is not an eigenvalue of $(L, \mathcal{B})$ we denote Green's function of $(L, \mathcal{B})$ by

$$(5.2) \quad G(x, t) = \begin{cases} b(x)a(t), & x \geq t, \\ a(x)b(t), & x \leq t, \end{cases}$$

where $ab' - a'b = 1/p$ as in chapter 2.

We note that we make no attempt to derive conditions for the existence and uniqueness of solutions of (2.1), being concerned here only with the numerical analysts problem of determining the condition under which a given numerical method will converge and its order of convergence. However it is well known that if the first eigenvalue μ_0 of the eigenvalue problem

$$(5.3) \quad \begin{cases} Ly + \mu y = 0, \\ B_1 y = B_2 y = 0, \end{cases}$$

is greater than zero and

$$e \geq f'_y(t, y) \geq \gamma > -\mu_0$$

for all $(t, y) \in [0, 1] \times \mathbb{R}$ then (5.1) has a unique solution.

Here we assume that $\gamma = 0$, i.e.

$$(5.4) \quad 0 \leq f'_y(t, y) \leq e, \quad (t, y) \in [0, 1] \times \mathbb{R}, \quad e > 0.$$

Throughout this chapter we will assume that $\mu_0 > 0$.

We have, already in chapter 3, noted the importance of the first eigenvalue λ_0 of the eigenvalue problem

$$(5.5) \quad \begin{cases} Ly + \lambda y = 0, \\ y(0) = y(1) = 0, \end{cases}$$

for the existence of the determinant formulation of (2.1). To show that $\mu_0 \leq \lambda_0$ we will need some comparison theorems for eigenfunctions and eigenvalues. Following Swanson [24] let $J(y)$ be the quadratic functional, associated with the eigenvalue problem (5.3) with $a_{12}b_{22} > 0$, defined by

$$J(y) = \int_0^1 [p(t)y'(t)^2 + q(t)y(t)^2] dt + p(1) \frac{b_{21}}{b_{22}} y(1) + p(0) \frac{a_{11}}{a_{12}} y(0),$$

with domain $\mathcal{D} = C^1[0,1]$. The analog of J in the case $a_{12} = 0$ and $b_{22} > 0$ is

$$J_0(y) = \int_0^1 [p(t)y'(t)^2 + q(t)y(t)^2] dt + p(1) \frac{b_{21}}{b_{22}} y(1)$$

whose domain $\mathcal{D}_0 = \{y \in \mathcal{D} \mid y(0) = 0\}$. Likewise the functionals $J_1(y)$ and $J_{01}(y)$ are defined by

$$J_1(y) = \int_0^1 [p(t)y'(t)^2 + q(t)y(t)^2] dt + p(0) \frac{a_{11}}{a_{12}} y(0),$$

$$J_{01}(y) = \int_0^1 [p(t)y'(t)^2 + q(t)y(t)^2] dt,$$

with domains $\mathcal{D}_1 = \{y \in \mathcal{D} \mid y(1) = 0\}$, $\mathcal{D}_{01} = \{y \in \mathcal{D} \mid y(0) = y(1) = 0\}$, respectively. The innerproduct of two functions $y_1, y_2 \in \mathcal{D}$ is defined as

$$(y_1, y_2) = \int_0^1 y_1(t)y_2(t) dt$$

and the norm of $y \in \mathcal{D}$ as

$$\|y\| = (y, y)^{\frac{1}{2}}.$$

By Courant's minimum principle we have:

A function $y_0 \in \mathcal{D}$ which minimizes $J(y)$ under the condition $\|y\| = 1$ is an eigenfunction (unique except for a constant factor of modulus 1) corresponding to the smallest eigenvalue $\mu_0 = J(y_0)$ of (5.3).

The same conclusion holds if $\mathcal{D}$ is replaced by $\mathcal{D}_0$, $\mathcal{D}_1$, or $\mathcal{D}_{01}$, $J(y)$ replaced by $J_0(y)$, $J_1(y)$, or $J_{01}(y)$, and the boundary conditions in (5.3) are replaced by $y(0) = B_2 y = 0$, $B_1 y = y(1) = 0$, or $y(0) = y(1) = 0$, respectively.

Lemma 5.1

We have $\mu_0 \leq \lambda_0$.

Proof: By Courant's minimum principle there is an eigenfunction

$y_0 \in \mathcal{D}_{01}$, $\|y_0\| = 1$, such that

$$\lambda_0 = J_{01}(y_0) = J(y_0)$$

since $y_0(0) = y_0(1) = 0$. It follows from the inclusion $\mathcal{D}_{01} \subset \mathcal{D}$ that

$$\mu_0 = \min_{\substack{y \in \mathcal{D} \\ \|y\|=1}} J(y) \leq J(y_0) = \lambda_0.$$

The same conclusion holds if $\mathcal{D}$ is replaced by $\mathcal{D}_0$ or $\mathcal{D}_1$.

If $\mathcal{D}$ is replaced by $\mathcal{D}_{01}$ we have $\mu_0 = \lambda_0$.

Remark.

By Lemma 5.1 the determinant formulation of the boundary value problem (2.1) with the eigenvalue $\mu_0 > 0$ is equivalent to the integral equation formulation on a given grid. This property also holds if $\mu_0 < 0$ and $\lambda_0 > 0$.

Lemma 5.2

The eigenvalue problem (5.3) has an infinite sequence of real eigenvalues $\mu_0 < \mu_1 < \mu_2 < \dots$ with $\lim_{n \rightarrow \infty} \mu_n = \infty$. The eigenfunction $z_n(x)$ belonging to the eigenvalue μ_n has exactly n zeros in the open interval $(0,1)$, and is uniquely determined up to a constant factor.

Proof: See Birkhoff and Rota [7] theorem 5 p. 296.

We now give a lemma concerning the zeros of the functions a and b in (5.2).

Lemma 5.3

If the first eigenvalue μ_0 of (5.3) is greater than zero then the function $a(x)$ has no zero in the interval $(0,1]$ and the function $b(x)$ has no zero in the interval $[0,1)$.

Proof: Suppose that $a(x_1) = 0$, $0 < x_1 \leq 1$. If $a_{12} = 0$ then $a(0) = 0$.

Let λ_0 be the first eigenvalue of

$$\begin{cases} Ly + \lambda y = 0, \\ y(0) = y(1) = 0, \end{cases}$$

and y_0 the eigenfunction belonging to the eigenvalue λ_0 . We have $\lambda_0 \geq \mu_0 > 0$. By Sturm's comparison theorem (Lemma 3.5) y_0 must have a zero between 0 and x_1 which contradict Lemma 5.2. Now assume that $a_{12} > 0$. Let z_0 be the eigenfunction belonging to the eigenvalue μ_0 of the eigenvalue problem (5.3). We have

$$La = 0,$$

$$Lz_0 + \mu_0 z_0 = 0,$$

which give

$$\frac{d}{dx} [p(az'_0 - a'z_0)] = -\mu_0 az_0.$$

Thus

$$(1) \quad \left[p(az'_0 - a'z_0) \right]_0^{x_1} = -\mu_0 \int_0^{x_1} az_0 dx,$$

where we assume that x_1 is the first zero of a in $(0,1]$. With no loss in generality a and z_0 may be regarded as positive within the interval $(0, x_1)$, since z_0 has no zero in the interval $(0,1)$ by Lemma 5.2. For the left hand side of (i) we have

$$\left[p(az'_0 - a'z_0) \right]_0^{x_1} = -p(x_1)a'(x_1)z_0(x_1).$$

Since $a(0) > 0$ and x_1 is the first zero of a we must have $a'(x_1) \leq 0$. Hence

$$\left[p(az'_0 - a'z_0) \right]_0^{x_1} \geq 0.$$

But for the right hand side of (i) we have

$$-\mu_0 \int_0^{x_1} az_0 dx < 0,$$

which leads to a contradiction. Hence a cannot have a zero in the interval $(0,1]$.

In the same way we can show that b cannot have a zero in the interval $[0,1)$.

Following Ortega [16] we turn next to some results which involve the idea of a partial ordering. For vectors $x, y \in \mathbb{R}^n$ we define the natural or componentwise partial ordering by

$$x \leq y \text{ if and only if } x_i \leq y_i, \quad i = 1, \dots, n$$

and similarly for matrices $A, B: \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$A \leq B \text{ if and only if } a_{ij} \leq b_{ij}, \quad i, j = 1, \dots, n.$$

In the special case that $x = 0$ or $A = 0$ we say that $y \geq 0$ or $B \geq 0$ is nonnegative. An important class of nonnegative vectors or matrices is given by the absolute values

$$|x| = (|x_1|, \dots, |x_n|^T), \quad |A| = (|a_{ij}|).$$

Definition

A $n \times n$ matrix A is an M-matrix if $A^{-1} \geq 0$ and $a_{i,j} \leq 0, i \neq j$. A symmetric M-matrix is a Stieltjes matrix.

Let $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$ be a grid in the interval $[0,1]$. We will now consider the matrix T defined by the left hand side of the determinant formulation (3.5), (3.7), (3.9) of the boundary value problem (2.1) with $\mu_0 > 0$.

Lemma 5.4

We have

$$(5.6) \quad D(x_i, x_{i+1}) < 0 \quad \text{for } i = 0, \dots, n-1.$$

Proof: Consider $D(x_i, t)$ for $x_i \leq t \leq 1$. We have $D(x_i, x_i) = 0$ and

$$\frac{d}{dt} D(x_i, t) = \begin{vmatrix} b(x_i) & a(x_i) \\ b'(t) & a'(t) \end{vmatrix} = -[a(x_i)b'(t) - a'(t)b(x_i)].$$

Thus

$$\left. \frac{d}{dt} D(x_i, t) \right|_{t=x_i} = -[a(x_i)b'(x_i) - a'(x_i)b(x_i)] = \frac{-1}{p(x_i)} < 0.$$

Hence $D(x_i, t) < 0$ to the right of x_i . If $D(x_i, x_{i+1}) \geq 0$ there must exist a t_0 such that $x_i < t_0 \leq x_{i+1}$ and $D(x_i, t_0) = 0$.

But $\lambda_0 \geq \mu_0 > 0$ and by Theorem 3.6 we must have $D(x_i, t_0) \neq 0$.

Thus $D(x_i, x_{i+1}) < 0$.

By Theorem 3.3 the matrix T is nonsingular and $T^{-1} = G = (g_{i,j})$ where

$$g_{i,j} = \begin{cases} b(x_i)a(x_j), & i \geq j, \\ a(x_i)b(x_j), & i \leq j, \end{cases}$$

and

$$i,j \geq \begin{cases} 0 & \text{if } a_{12} > 0, \\ 1 & \text{if } a_{12} = 0, \end{cases} \quad i,j \leq \begin{cases} n & \text{if } b_{22} > 0, \\ n-1 & \text{if } b_{22} = 0. \end{cases}$$

Theorem 5.5

The matrix $-T$ is a Stieltjes matrix.

Proof: Let $T = (t_{i,j})$. $-T$ is symmetric and $1/D(x_{i-1}, x_i) < 0$, $i = 1, \dots, n$. Hence $-t_{i,j} \leq 0$, $i \neq j$. We have $T^{-1} = G = (g_{i,j})$ where $g_{i,j}$ is defined above. By Lemma 5.3 $a(x)$ is of one sign in the interval $[0, 1]$ and so is $b(x)$. Hence $G(x, t) \geq 0$ or $G(x, t) \leq 0$ in $[0, 1] \times [0, 1]$. But since all eigenvalues $0 < \mu_0 < \mu_1 < \dots$ of the eigenvalue problem

$$y(x) + \mu \int_0^1 G(x, t)y(t)dt = 0$$

are greater than zero we must have $G(x, t) \leq 0$. Hence $-g_{i,j} \geq 0$ and $-T$ is a symmetric M-matrix.

Remark.

Since a Stieltjes matrix is positive definite the matrix $-T$ is also positive definite.

Proposition 5.6

Let A be an M-matrix and D be a nonnegative diagonal matrix.

Then $A + D$ is an M-matrix and

$$(5.7) \quad (A + D)^{-1} \leq A^{-1}.$$

Proof: (See Ortega [16] p. 109.)

Let $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$. For simplicity we first assume that $a_{12}b_{22} > 0$. Discretization of the determinant formulation of the boundary value problem (2.1) by the trapezoidal rule is given by (4.1). Put

$$(5.8) \quad h = \max_{i=1, \dots, n} (x_i - x_{i-1}).$$

In vector notation we have by (4.1)

$$(5.9) \quad Ty = Wf(y) + \beta - \tau(h),$$

where $y = (y(0), y(x_1), \dots, y(1))^T$, $f(y) = (f(0, y(0)), f(x_1, y(x_1)), \dots, f(1, y(1)))^T$, $\tau(h) = (\tau_0(h), \tau_1(h), \dots, \tau_n(h))^T$, $\beta = (\frac{c}{a(0)B_1(b)}, 0, \dots, 0, \frac{d}{b(1)B_2(a)})^T$, $y(x)$ is the exact solution of the boundary value problem, and W is the diagonal matrix $W = \text{diag}(w_0, w_1, \dots, w_n)$,

$$w_i = \begin{cases} \frac{x_1}{2}, & i = 0, \\ \frac{x_{i+1} - x_{i-1}}{2}, & i = 1, \dots, n-1, \\ \frac{1 - x_{n-1}}{2}, & i = n. \end{cases}$$

$\tau_i(h)$, $i = 0, \dots, n$, represent the local truncation errors caused by the approximation of the integrals by the trapezoidal rule. We assume that $f(x, y)$ is twice continuous differentiable on $[0, 1] \times R$. Put

$$F_i(t) = \Delta_i(t)f(t, y(t)), \quad t \in \begin{cases} [0, x_1], & \text{if } i = 0, \\ [x_{i-1}, x_{i+1}], & \text{if } i = 1, \dots, n-1, \\ [x_{n-1}, 1], & \text{if } i = n. \end{cases}$$

We have $F_0 \in C^2[0, x_1]$, $F_i \in C^2[x_{i-1}, x_i]$ and $F_i \in C^2[x_i, x_{i+1}]$ for $i = 1, \dots, n-1$, $F_n \in C^2[x_{n-1}, 1]$. Put

$$(5.10) \quad \begin{cases} M_0 = \max_{0 \leq t \leq x_1} |F_0''(t)|, \\ M_i = \max \left\{ \max_{x_{i-1} \leq t \leq x_i} |F_i''(t)|, \max_{x_i \leq t \leq x_{i+1}} |F_i''(t)| \right\}, & i = 1, \dots, n-1, \\ M_n = \max_{x_{n-1} \leq t \leq 1} |F_n''(t)|, \end{cases}$$

and

$$(5.11) \quad M = \max_{i=0, \dots, n} M_i.$$

We have

$$(5.12) \quad \tau_i(h) = \begin{cases} \frac{x_1^3}{12} F_0''(t_0), & 0 < t_0 < x_1, \quad i = 0, \\ \frac{(x_i - x_{i-1})^3}{12} F_1''(t_1^-) + \frac{(x_{i+1} - x_i)^3}{12} F_1''(t_1^+), \\ x_{i-1} < t_1^- < x_i, \quad x_i < t_1^+ < x_{i+1}, \quad i = 1, \dots, n-1, \\ \frac{(1 - x_{n-1})^3}{12} F_n''(t_n), & x_{n-1} < t_n < 1, \quad i = n. \end{cases}$$

Hence

$$(5.13) \quad |\tau_i(h)| < w_i \frac{M}{2} h^2, \quad i = 0, \dots, n.$$

Let $Y = (y_0, y_1, \dots, y_n)^T$ be the solution of (5.9) with the local truncation error set to zero. Then we have

$$(5.14) \quad T(y - Y) = W(f(y) - f(Y)) - \tau(h).$$

By the Mean Value Theorem we have

$$f(x_i, y(x_i)) - f(x_i, y_i) = f'_y(x_i, \eta_i)(y(x_i) - y_i)$$

for some η_i between $y(x_i)$ and y_i , $i = 0, \dots, n$. Let D be the diagonal matrix $D = \text{diag}(f'_y(x_0, \eta_0), \dots, f'_y(x_n, \eta_n))$. By (5.4) D is nonnegative. Now (5.14) can be written as

$$(-T + WD)(y - Y) = \tau(h).$$

By Theorem 5.5 and Proposition 5.6 the matrix $-T + WD$ is an M -matrix since WD is a nonnegative diagonal matrix. Hence we have

$$0 \leq (-T + WD)^{-1} \leq -T^{-1} = -G.$$

By using the absolute values we get

$$|y - Y| \leq -G |\tau(h)| = -GW(W^{-1} |\tau(h)|).$$

Since the element in $-G$ are nonnegative we get

$$|y(x_i) - y_i| \leq \sum_j w_j |g_{i,j}| \frac{|\tau_j(h)|}{w_j}, \quad i = 0, \dots, n.$$

Let

$$\max_{0 \leq x, t \leq 1} |G(x, t)| = C.$$

Then by (5.13) we get

$$|y(x_i) - y_i| \leq \frac{CM}{2} h^2, \quad i = 0, \dots, n.$$

We have shown the following theorem.

Theorem 5.7

Let $\mu_0 > 0$ for the boundary value problem (2.1) and let f be twice continuous differentiable in $[0, 1] \times R$. Suppose in addition that

$$f'_y(x, y) \geq 0 \quad \text{in} \quad [0, 1] \times R.$$

Then, with y_i , $i = 0, \dots, n$, the solution of (4.1), we have

$$(5.15) \quad \max_{i=0, \dots, n} |y(x_i) - y_i| = O(h^2),$$

where $y(x)$ is the exact solution of (2.1) and h is defined by (5.8).

Theorem 5.8

The system

$$TY = Wf(Y) + \beta$$

has a unique solution.

Proof: By theorem 4.4.1 in Ortega and Rheinboldt [17] the mapping $K: R^{n+1} \rightarrow R^{n+1}$ defined by $KY = -TY + Wf(Y)$ is a homeomorphism of R^{n+1} onto R^{n+1} .

Remark.

If $a_{12} = b_{22} = 0$ then $\mu_0 = \lambda_0$, $\beta = \left(\frac{c}{a_{11}D(0, x_1)}, 0, \dots, 0, \frac{d}{b_{21}D(x_{n-1}, 1)} \right)^T$.

Hence the Theorems 5.7 and 5.8 apply to this case. The Theorems also apply if only one of a_{12} or b_{22} is zero.

The discrete determinant formulation (4.1) only give the solution on the discrete set $x_i, i = 0, \dots, n$. The discretization of the integral equation (2.5) by the trapezoidal rule is

$$(5.16) \quad \hat{y}(x) = \sum_{j=0}^n w_j G(x, x_j) f(x_j, y_j) + \frac{cb(x)}{B_1(b)} + \frac{da(x)}{B_2(a)}, \quad 0 \leq x \leq 1,$$

where $\hat{y}(x_j) = y_j, j = 0, \dots, n$. (5.16) can be used as an interpolation formula. It is often called the "natural" interpolation formula.

The determinant formulation of (5.16) on x_i, x, x_{i+1} , where $x_i \leq x \leq x_{i+1}$, is

$$\begin{vmatrix} b(x_i) & a(x_i) & y_i \\ b(x) & a(x) & \hat{y}(x) \\ b(x_{i+1}) & a(x_{i+1}) & y_{i+1} \end{vmatrix} = 0$$

or

$$(5.17) \quad \hat{y}(x) = \frac{D(x, x_{i+1})}{D(x_i, x_{i+1})} y_i + \frac{D(x_i, x)}{D(x_i, x_{i+1})} y_{i+1}, \quad x_i \leq x \leq x_{i+1},$$

for $i = 0, 1, \dots, n-1$. We have

$$\begin{aligned} y(x) - \hat{y}(x) &= y(x) - \frac{D(x, x_{i+1})}{D(x_i, x_{i+1})} y(x_i) - \frac{D(x_i, x)}{D(x_i, x_{i+1})} y(x_{i+1}) + \\ &+ \frac{D(x, x_{i+1})}{D(x_i, x_{i+1})} [y(x_i) - y_i] + \frac{D(x_i, x)}{D(x_i, x_{i+1})} [y(x_{i+1}) - y_{i+1}], \end{aligned}$$

for $x_i \leq x \leq x_{i+1}, i = 0, 1, \dots, n-1$.

By Theorem 5.7 we have

$$\frac{D(x, x_{i+1})}{D(x_i, x_{i+1})} |y(x_i) - y_i| + \frac{D(x_i, x)}{D(x_i, x_{i+1})} |y(x_{i+1}) - y_{i+1}| = o(h^2).$$

By the determinant formulation of the integral equation (2.5) we have

$$\begin{aligned}
 y(x) - \frac{D(x, x_{i+1})}{D(x_i, x_{i+1})} y(x_i) - \frac{D(x_i, x)}{D(x_i, x_{i+1})} y(x_{i+1}) &= \\
 = D(x, x_{i+1}) D(x_i, x) \frac{1}{D(x_i, x_{i+1})} \int_{x_i}^{x_{i+1}} \Delta(t) f(t, y(t)) dt
 \end{aligned}$$

where

$$\Delta(t) = \begin{cases} \frac{D(x_i, t)}{D(x_i, x)}, & x_i \leq t \leq x, \\ \frac{D(t, x_{i+1})}{D(x, x_{i+1})}, & x \leq t \leq x_{i+1}. \end{cases}$$

We have

$$\frac{1}{D(x_i, x_{i+1})} \int_{x_i}^{x_{i+1}} \Delta(t) f(t, y(t)) dt = O(1)$$

and

$$D(x, x_{i+1}) D(x_i, x) = (x_{i+1} - x)(x - x_i) p(x)^2 + O(h^3).$$

Thus

$$(5.18) \quad \max_{0 \leq x \leq 1} |y(x) - \hat{y}(x)| = O(h^2).$$

We will now consider the discretization (4.7), (4.15), and (4.16) of the determinant formulation of the boundary value problem (2.1).

We assume that $p \in C^3[0, 1]$, $q \in C^2[0, 1]$, and $f \in C^4([0, 1] \times \mathbb{R})$.

Then $y \in C^4[0, 1]$. The grid is $x_i = i/n$, $i = 0, \dots, n$, $h = 1/n$.

In vector notation (4.7), (4.15), and (4.16) (we assume that $a_{12} b_{22} > 0$) can be written as

$$(5.19) \quad TY = h\tilde{I}f(Y) + \frac{h^2}{12} TC_p f(Y) + hBf(Y) + \beta,$$

where $\tilde{I} = \text{diag}(\frac{1}{3}, 1, \dots, 1, \frac{1}{3})$, $C_p = \text{diag}(\frac{1}{p(0)}, \frac{1}{p(1/n)}, \dots, \frac{1}{p(1)})$, T is

the tridiagonal matrix defined by the left hand side of (4.7), (4.15), and (4.16), the matrix B have zero entries except for the first and last row which are

$$(5.20) \quad \begin{cases} \left(\frac{-ha(\frac{2}{n})}{12a(0)D(0,\frac{2}{n})p(0)} \quad \frac{D(\frac{1}{n},\frac{2}{n})}{3D(0,\frac{2}{n})} \quad \frac{1}{12D(0,\frac{2}{n})p(\frac{2}{n})} \quad 0 \quad \dots \quad 0 \right) \\ \left(0 \quad \dots \quad 0 \quad \frac{1}{12D(\frac{n-2}{n},1)p(\frac{n-2}{n})} \quad \frac{D(\frac{n-2}{n},\frac{n-1}{n})}{3D(\frac{n-2}{n},1)} \quad \frac{-hb(\frac{n-2}{n})}{12b(1)D(\frac{n-2}{n},1)p(1)} \right), \end{cases}$$

$$\beta = \left(\frac{c}{a(0)B_1(b)}, 0, \dots, 0, \frac{d}{b(1)B_2(a)} \right)^T, \quad Y = (y_0, y_1, \dots, y_n)^T.$$

We also have

$$(5.21) \quad Ty = h\tilde{I}f(y) + \frac{h^2}{12} TC_p f(y) + hBf(y) + \beta + \tau(h),$$

where $y = (y(0), y(\frac{1}{n}), \dots, y(1))^T$ is the exact solution of the boundary value problem (2.1) and $\tau(h) = (\tau_0(h), \tau_1(h), \dots, \tau_n(h))^T$ represents the local truncation error caused by the approximation of the integrals by the trapezoidal rule with "end correction" and Simpson's rule.

Put

$$\begin{cases} F_0(t) = D(t, \frac{2}{n})f(t, y(t))/D(0, \frac{2}{n}), & 0 \leq t \leq \frac{2}{n}, \\ F_i(t) = \Delta_i(t)f(t, y(t)), & \frac{i-1}{n} \leq t \leq \frac{i+1}{n}, \quad i = 1, \dots, n-1, \\ F_n(t) = D(\frac{n-2}{n}, t)f(t, y(t))/D(\frac{n-2}{n}, 1), & \frac{n-2}{n} \leq t \leq 1. \end{cases}$$

$$\begin{cases} M_0 = \max_{0 \leq t \leq \frac{2}{n}} |F_0^{(4)}(t)|, \\ M_i = \max \left\{ \max_{\frac{i-1}{n} \leq t \leq \frac{i}{n}} |F_i^{(4)}(t)|, \max_{\frac{i}{n} \leq t \leq \frac{i+1}{n}} |F_i^{(4)}(t)| \right\}, \quad i = 1, \dots, n-1, \\ M_n = \max_{\frac{n-2}{n} \leq t \leq 1} |F_n^{(4)}(t)|, \end{cases}$$

and

$$(5.22) \quad M = \max_{i=0, \dots, n} M_i.$$

The local truncation error for the trapezoidal rule with "end correction" is $\frac{h^5}{720} F^{(4)}(\xi)$, we get

$$(5.23) \quad \tau_i(h) = \frac{h^5}{720} F_i^{(4)}(t_i^-) + \frac{h^5}{720} F_i^{(4)}(t_i^+), \quad \frac{i-1}{n} < t_i^- < \frac{i}{n}, \quad \frac{i}{n} < t_i^+ < \frac{i+1}{n},$$

for $i = 1, \dots, n-1$. The local truncation error for Simpson's rule is $-\frac{h^5}{90} F^{(4)}(\xi)$. By (4.13) and (4.14) we get

$$(5.24) \quad \tau_0(h) = -\frac{h^5}{90} F_0^{(4)}(t_0) - \frac{D(\frac{1}{n}, \frac{2}{n})}{D(0, \frac{2}{n})} \tau_1(h), \quad 0 < t_0 < \frac{2}{n}.$$

In the same way we get

$$(5.25) \quad \tau_n(h) = -\frac{h^5}{90} F_0^{(4)}(t_n) - \frac{D(\frac{n-2}{n}, \frac{n-1}{n})}{D(\frac{n-2}{n}, 1)} \tau_{n-1}(h), \quad \frac{n-2}{n} < t_n < 1.$$

Since

$$0 < \frac{D(\frac{n-2}{n}, \frac{n-1}{n})}{D(\frac{n-2}{n}, 1)} \sim \frac{1}{2}$$

we may assume that $D(\frac{n-2}{n}, \frac{n-1}{n})/D(\frac{n-2}{n}, 1) < 1$ for h sufficiently

small. Hence

$$(5.26) \quad |\tau_i(h)| < \frac{M}{72} h^5, \quad i = 0, \dots, n,$$

for h sufficiently small.

By subtracting (5.19) and (5.21) we get

$$(5.27) \quad T(y - Y) = h\tilde{T}[f(y) - f(Y)] + \frac{h^2}{12} TC_p[f(y) - f(Y)] + \\ + hB[f(y) - f(Y)] + \tau(h).$$

By the Mean Value Theorem we have

$$f(\frac{i}{n}, y(\frac{i}{n})) - f(\frac{i}{n}, y_i) = f'_y(\frac{i}{n}, \eta_i)[y(\frac{i}{n}) - y_i],$$

for some η_i between $y(\frac{i}{n})$ and y_i , $i = 0, \dots, n$, where by (5.4)

$$(5.28) \quad 0 \leq f'_y(\frac{i}{n}, \eta_i) \leq e.$$

Let D be the diagonal matrix $D = \text{diag}(f'_y(0, \eta_0), \dots, f'_y(1, \eta_n))$.

By (5.28) D is nonnegative. (5.27) can be written as

$$(5.29) \quad \left[T(I - \frac{h^2}{12} C_p D) - h\tilde{I}D \right] (y - Y) = hBD(y - Y) + \tau(h).$$

Consider the matrix

$$T(I - \frac{h^2}{12} C_p D).$$

If

$$n^2 > \frac{e}{12 \min_{0 \leq t \leq 1} p(t)}$$

then $I - \frac{h^2}{12} C_p D$ is a diagonal matrix with strictly positive diagonal element. Thus $-T(I - \frac{h^2}{12} C_p D)$ is an M-matrix, and by Proposition 5.6 the matrix

$$-T(I - \frac{h^2}{12} C_p D) + h\tilde{I}D$$

is an M-matrix and

$$(5.30) \quad [-T(I - \frac{h^2}{12} C_p D) + h\tilde{I}D]^{-1} \leq (I - \frac{h^2}{12} C_p D)^{-1} (-T^{-1}).$$

Thus

$$\begin{aligned} |y - Y| &\leq (I - \frac{h^2}{12} C_p D)^{-1} (-T^{-1}) h |B| D |y - Y| + \\ &+ (I - \frac{h^2}{12} C_p D)^{-1} (-T^{-1}) |\tau(h)|. \end{aligned}$$

Consider the matrix $-hT^{-1}$, as above we have

$$\| -hT^{-1} \|_{\infty} \leq C$$

where $C = \max_{0 \leq x, t \leq 1} |G(x, t)|$. Put

$$A = \frac{1}{1 - \frac{h^2 e}{12 \min_{0 \leq t \leq 1} p(t)}}$$

then

$$\| (I - \frac{h^2}{12} C_p D)^{-1} \|_{\infty} \leq A$$

for h sufficiently small. We also have

$$|B| \sim \frac{1}{24} \begin{pmatrix} 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 4 & 1 \end{pmatrix}$$

for h sufficiently small. Hence

$$\| -T^{-1} |B| \|_{\infty} \leq 2C \frac{6}{24} (1 + O(h)) \leq C$$

for h sufficiently small. Thus

$$\|y - Y\|_{\infty} \leq hACe \|y - Y\|_{\infty} + AC \frac{M}{72} h^4$$

and

$$(5.31) \quad \|y - Y\|_{\infty} \leq \frac{ACM}{1 - hACe} \frac{1}{72} h^4$$

for h sufficiently small.

We have shown the following theorem.

Theorem 5.9

Let $\mu_0 > 0$ for the boundary value problem (2.1) and let $p \in C^3[0,1]$, $q \in C^2[0,1]$ and $f \in C^4([0,1] \times \mathbb{R})$. Suppose in addition that

$$0 \leq f'_y(x,y) \leq e, \quad \text{on } [0,1] \times \mathbb{R}.$$

Then, with y_i , $i = 0, \dots, n$, the solution of (4.7), (4.15), and (4.16), we have

$$(5.32) \quad \max_{i=0, \dots, n} |y(x_i) - y_i| = O(h^4),$$

where $y(x)$ is the exact solution of (2.1).

Theorem 5.10

The system

$$(5.33) \quad TY = h\tilde{I}f(Y) + \frac{h^2}{12} TC_p f(Y) + hBf(Y) + \beta$$

has a solution for h sufficiently small.

Proof: Consider (5.33), we have

$$\begin{aligned} -TY + \frac{h^2}{12} TC_p [f(Y) - f(0)] + h\tilde{I}[f(Y) - f(0)] &= \\ = -h \left[\tilde{I} + \frac{h}{12} TC_p \right] f(0) - hBf(Y) - \beta. \end{aligned}$$

Since $f(Y) - f(0) = D(Y)Y$, where

$$D(Y) = \text{diag} \left(\int_0^1 f'_y(0, ty_0) dt, \dots, \int_0^1 f'_y(1, ty_n) dt \right)$$

we can write

$$(5.34) \quad -TY + \frac{h^2}{12} TC_p D(Y)Y + h\tilde{I}D(Y)Y = -h \left[\tilde{I} + \frac{h}{12} TC_p \right] f(0) - hBD(Y)Y - hBf(0) - \beta.$$

(5.34) is equivalent to the system

$$\begin{aligned} Y = \left[-T + \frac{h^2}{12} TC_p D(Y) + h\tilde{I}D(Y) \right]^{-1} \left\{ -h \left[\tilde{I} + \frac{h}{12} TC_p \right] f(0) - \right. \\ \left. - hBD(Y)Y - hBf(0) - \beta \right\} = F(Y). \end{aligned}$$

Then as in the proof of Theorem 5.9 we have

$$\begin{aligned} |F(Y)| \leq \left[I - \frac{h^2}{12} C_p D(Y) \right]^{-1} (-T^{-1}) \left\{ h \left| \tilde{I} + \frac{h}{12} TC_p \right| |f(0)| + \right. \\ \left. + h|B| |D(Y)| |Y| + h|B| |f(0)| + |\beta| \right\}. \end{aligned}$$

We have

$$\tilde{I} + \frac{h}{12} TC_p \sim \frac{1}{12} \begin{bmatrix} 3 & 1 & & 0 \\ 1 & 10 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & 10 & 1 \\ 0 & & & 1 & 3 \end{bmatrix}$$

and

$$\|\tilde{I} + \frac{h}{12} TC_p\|_{\infty} = 1 + O(h) < 2$$

if h is sufficiently small. The elements in the vector $-T^{-1}\beta$ are

$$\begin{aligned} & -\varepsilon_{i,0} \frac{c}{a(0)B_1(b)} - \varepsilon_{i,n} \frac{d}{b(1)B_2(a)} = \\ & = \frac{b(\frac{1}{n})a(0)p(0)c}{a_{12}} + \frac{a(\frac{1}{n})b(1)p(1)d}{b_{22}}, \quad i = 0, \dots, n, \end{aligned}$$

by (2.6) and (2.7). Hence

$$\|T^{-1}\beta\|_{\infty} \leq 2C\|\beta\|_{\infty}.$$

We get

$$\begin{aligned} \|F(Y)\|_{\infty} & \leq 2AC\|f(0)\|_{\infty} + hACe\|Y\|_{\infty} + \\ & + hAC\|f(0)\|_{\infty} + 2AC\|\beta\|_{\infty} \end{aligned}$$

for h sufficiently small. Hence if

$$\|Y\|_{\infty} \leq \frac{AC[(2+h)\|f(0)\|_{\infty} + 2\|\beta\|_{\infty}]}{1 - hACe} = r$$

we get

$$\|F(Y)\|_{\infty} \leq r$$

for h sufficiently small. Thus, the sphere $S = \{z \mid \|z\|_{\infty} \leq r\}$

is mapped into itself by F ; and the conclusion follows by the

Brouwer fixed point theorem.

Remark.

If $a_{12} = b_{22} = 0$ then $\mu_0 = \lambda_0$, $B = 0$, $\tilde{I} = I$, and

$$\beta = \begin{bmatrix} \frac{1}{D(\frac{1}{n}, \frac{1}{n})} \left(\frac{c}{a_{11}} - \frac{h^2}{12} \frac{f(0, c/a_{11})}{p(0)} \right) \\ 0 \\ \vdots \\ 0 \\ \frac{1}{D(\frac{n-1}{n}, 1)} \left(\frac{d}{b_{21}} - \frac{h^2}{12} \frac{f(1, d/b_{21})}{p(1)} \right) \end{bmatrix}.$$

Hence the Theorems 5.9 and 5.10 apply to this case. The Theorems also apply if only one of a_{12} or b_{22} is zero.

We will now consider the discretization (4.36) of the determinant formulation of the boundary value problem (2.8). In (4.36)

$D(x_{i-1}, x_i)$ have been approximated by the midpoint rule

$$-D(x_{i-1}, x_i) = \int_{x_{i-1}}^{x_i} \frac{dt}{p(t)} = \frac{x_i - x_{i-1}}{p(\frac{x_{i-1} + x_i}{2})} + \frac{(x_i - x_{i-1})^3}{4} \frac{d^2}{dt^2} \frac{1}{p(t)} \Big|_{t=\xi},$$

$x_{i-1} < \xi < x_i$, for $i = 1, \dots, n$. Let

$$-\hat{D}(x_{i-1}, x_i) = \frac{x_i - x_{i-1}}{p(\frac{x_{i-1} + x_i}{2})}, \quad i = 1, \dots, n,$$

and let $\hat{T}$ be the matrix T , defined by the left hand side of (4.35),

where $D(x_{i-1}, x_i)$, $i = 1, \dots, n$, have been replaced by $\hat{D}(x_{i-1}, x_i)$,

$i = 1, \dots, n$. It is easy to see that $\hat{T}^{-1} = \hat{G} = (\hat{g}_{i,j})$, where

$$\hat{g}_{i,j} = \begin{cases} \hat{b}(x_i) \hat{a}(x_j), & i \geq j, \\ \hat{a}(x_i) \hat{b}(x_j), & i \leq j, \end{cases}$$

and

$$i, j \geq \begin{cases} 0 & \text{if } a_{12} > 0, \\ 1 & \text{if } a_{12} = 0, \end{cases} \quad i, j \leq \begin{cases} n & \text{if } b_{22} > 0, \\ n-1 & \text{if } b_{22} = 0, \end{cases}$$

$$\begin{cases} \hat{a}(x_i) = \frac{a_{11}p(0)\hat{P}(x_i) + a_{12}}{D_p}, \\ \hat{b}(x_i) = b_{21}p(1)[\hat{P}(x_i) - \hat{P}(1)] - b_{22}, \end{cases}$$

for $i = 0, \dots, n$, where

$$\hat{P}(x_i) = \begin{cases} 0, & \text{for } i = 0, \\ - \sum_{j=0}^{i-1} \frac{x_{j+1} - x_j}{p(\frac{x_j + x_{j+1}}{2})}, & \text{for } i = 1, \dots, n, \end{cases}$$

$$\hat{D}_p = a_{11}b_{21}p(0)p(1)\hat{P}(1) + a_{11}b_{22}p(0) + a_{12}b_{21}p(1),$$

i.e. $\int_0^{x_i} dt/p(t)$ has been replaced by the compound midpoint rule.

We note that $-\hat{T}$ is a Stieltjes matrix.

Put

$$(5.35) \quad h = \max_{i=1, \dots, n} (x_i - x_{i-1}).$$

Let y be the exact solution of the boundary value problem (2.8).

Then by (4.36) we have

$$(5.36) \quad \hat{T}y = Wf(y) - \beta + \hat{\tau}(h),$$

where $\hat{\tau}(h) = (\hat{\tau}_0(h), \hat{\tau}_1(h), \dots, \hat{\tau}_n(h))^T$ represents the local truncation error and $\beta = (cp(0)/a_{12}, 0, \dots, 0, dp(1)/b_{22})^T$ (we assume that $a_{12}b_{22} > 0$). Let $f \in C^2([0,1] \times \mathbb{R})$, $p \in C^3[0,1]$ then we have the following theorem.

Theorem 5.11

If $x_i = \frac{i}{n}$, $i = 0, \dots, n$ we have

$$\hat{\tau}_i(h) = \begin{cases} o(h^2), & \text{for } i = 0, n, \\ o(h^3), & \text{for } i = 1, \dots, n-1. \end{cases}$$

If the grid is not equidistant we have

$$\hat{\tau}_i(h) = o(h^2), \quad i = 0, \dots, n.$$

Proof: Consider equation (4.36). By Taylor's theorem we have

$$\begin{aligned}
& - \left[\frac{p(\frac{x_1}{2})}{x_1} + \frac{a_{11}}{a_{12}} p(0) \right] y(0) + \frac{p(\frac{x_1}{2})}{x_1} y(x_1) - \frac{x_1}{2} f(0, y(0)) + \\
& + \frac{cp(0)}{a_{12}} = x_1^2 \left[\frac{p''(0)y'(0)}{8} + \frac{p'(0)y''(0)}{4} \right] + o(x_1^3).
\end{aligned}$$

Hence $\hat{\tau}_0(h) = o(h^2)$ and by symmetry we also have $\hat{\tau}_n(h) = o(h^2)$.

For the other equations in (4.36) Taylor's theorem give

$$\begin{aligned}
& \frac{p(\frac{x_{i-1}+x_i}{2})}{x_i - x_{i-1}} [y(x_{i-1}) - y(x_i)] - \frac{p(\frac{x_i+x_{i+1}}{2})}{x_{i+1} - x_i} [y(x_i) - y(x_{i+1})] - \\
& - \frac{x_{i+1} - x_{i-1}}{2} f(x_i, y(x_i)) = \\
& = \frac{(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2}{96} [3p''(x_i)y'(x_i) + \\
& + 6p'(x_i)y''(x_i) + 4p(x_i)y'''(x_i)] + \\
& + \frac{(x_i - x_{i-1})^3 + (x_{i+1} - x_i)^3}{384} [p'''(x_i)y'(x_i) + \\
& + 3p''(x_i)y''(x_i) + 4p'(x_i)y'''(x_i) + 2p(x_i)y^{(4)}(x_i)] + o(h^4).
\end{aligned}$$

Hence $\hat{\tau}_i(h) = o(h^3)$ if $x_i = \frac{i}{n}$, $i = 0, \dots, n$ and $\hat{\tau}_i(h) = o(h^2)$ otherwise.

Let $x_i = \frac{i}{n}$, $i = 0, \dots, n$, $|\hat{g}_{i,j}| \leq C$ and consider a row of $-\hat{G}|\tau(h)|$, we have

$$\begin{aligned}
& -\hat{g}_{i,0}|\hat{\tau}_0(h)| + \sum_{j=1}^{n-1} -\hat{g}_{i,j}|\hat{\tau}_j(h)| - \hat{g}_{i,n}|\hat{\tau}_n(h)| \leq \\
& \leq CMh^2(1 + \frac{n-1}{n} + 1) \leq 3CMh^2,
\end{aligned}$$

where M is a constant independent of h . Hence

$$||-\hat{G}|\hat{\tau}(h)||_{\infty} = o(h^2).$$

Let $Y = (y_0, y_1, \dots, y_n)^T$ be the solution of (5.36) with $\hat{\tau}(h) = 0$.

By Theorem 5.7 we have

$$(5.37) \quad \max_{i=0, \dots, n} \left| y\left(\frac{i}{n}\right) - y_i \right| = O(h^2)$$

if $f'_y(x, y) \geq 0$ on $[0, 1] \times R$. As for the Remark after Theorem 5.7

(5.37) also holds for $a_{12} = 0$ or $b_{22} = 0$. It is also clear that Theorem 5.8 applies.

We will now consider the discretization (4.38), (4.39), (4.40)

of the boundary value problem (2.8). We will replace the integral

$$\int_{\frac{i-1}{n}}^{\frac{i}{n}} dt/p(t) \text{ by Simpson's rule. Hence put}$$

$$- \hat{D}\left(\frac{i-1}{n}, \frac{i}{n}\right) = \frac{h}{6} \left[\frac{1}{p\left(\frac{i-1}{n}\right)} + \frac{4}{p\left(\frac{i-1/2}{n}\right)} + \frac{1}{p\left(\frac{i}{n}\right)} \right].$$

Then we have $(a_{12} b_{22} > 0)$

$$(5.38) \quad \hat{T}y = h\tilde{I}f(y) + \frac{h^2}{12} \hat{T}C_p f(y) + h\hat{B}f(y) + \beta + \hat{\tau}(h),$$

where $\hat{T}$ and $\hat{B}$ are the same matrices as in equation (5.19) with

$D\left(\frac{i-1}{n}, \frac{i}{n}\right)$ replaced by $\hat{D}\left(\frac{i-1}{n}, \frac{i}{n}\right)$, $\beta = \left(\frac{cp(0)}{a_{12}}, 0, \dots, 0, \frac{dp(1)}{b_{22}}\right)^T$, and

$y = (y(0), y(\frac{1}{n}), \dots, y(1))^T$ is the exact solution of the boundary value problem (2.8).

Theorem 5.12

We have

$$\hat{\tau}_i(h) = \begin{cases} O(h^4) & \text{for } i = 0, n, \\ O(h^5) & \text{for } i = 1, \dots, n-1, \end{cases}$$

if $p \in C^5[0, 1]$, $f \in C^4([0, 1] \times R)$.

Proof: Consider equation (4.39). By Taylor's theorem we have

$$\begin{aligned}
& - \frac{1}{\hat{D}(\frac{i-1}{n}, \frac{i}{n})} \left\{ y(\frac{i}{n}) - y(\frac{i-1}{n}) - \frac{h^2}{12} \left[\frac{f(\frac{i}{n}, y(\frac{i}{n}))}{p(\frac{i}{n})} - \frac{f(\frac{i-1}{n}, y(\frac{i-1}{n}))}{p(\frac{i-1}{n})} \right] \right\} - \\
& - \frac{1}{\hat{D}(\frac{i}{n}, \frac{i+1}{n})} \left\{ y(\frac{i+1}{n}) - y(\frac{i}{n}) - \frac{h^2}{12} \left[\frac{f(\frac{i+1}{n}, y(\frac{i+1}{n}))}{p(\frac{i+1}{n})} - \frac{f(\frac{i}{n}, y(\frac{i}{n}))}{p(\frac{i}{n})} \right] \right\} - \\
& - hf(\frac{i}{n}, y(\frac{i}{n})) = \\
& = h^5 \left\{ - \frac{7}{720} y^{(6)} - \frac{1}{144} \frac{d^4}{dx^4} \left(\frac{p'y'}{p} \right) - \frac{p'}{180} y^{(5)} - \frac{p'}{72} \frac{d^3}{dx^3} \left(\frac{p'y'}{p} \right) - \right. \\
& - \frac{p}{4} \left[\frac{1}{18} \frac{p'''}{p} - \frac{1}{36} \left(\frac{p'}{p} \right)^2 \right] \frac{d^2}{dx^2} \left(\frac{p'y'}{p} \right) - \\
& \left. - \frac{p}{6} \left[\frac{1}{24} \frac{p''''}{p} - \frac{1}{12} \frac{p'p''}{p^2} + \frac{1}{24} \left(\frac{p'}{p} \right)^3 \right] \left[-y^{(3)} + \frac{d}{dx} \left(\frac{p'y'}{p} \right) \right] \right\} + o(h^6),
\end{aligned}$$

where all functions and their derivatives have been evaluated at the point $x = \frac{i}{n}$.

Now instead of equation (4.39) we consider that equation which corresponds to equation (4.13)

$$\begin{aligned}
& \left[\frac{1}{\hat{D}(0, \frac{2}{n})} - \frac{a_{11}p(0)}{a_{12}} \right] y_0 - \frac{y_2}{\hat{D}(0, \frac{2}{n})} = \frac{h}{3} \left[f(0, y_0) + 4 \frac{\hat{D}(\frac{1}{n}, \frac{2}{n})}{\hat{D}(0, \frac{2}{n})} f(\frac{1}{n}, y_1) \right] - \\
& - \frac{cp(0)}{a_{12}}.
\end{aligned}$$

By Taylor's theorem we have

$$\begin{aligned}
& \frac{1}{\hat{D}(0, \frac{2}{n})} = - \frac{6}{h} \left\{ \frac{p}{12} + \frac{p'}{12}h + \frac{1}{36} \left[-2 \frac{p'^2}{p} + p'' \right] h^2 + \right. \\
& + \frac{1}{36} \left[\frac{p'^3}{p^2} - 2 \frac{p'p''}{p} + p''' \right] h^3 + \frac{1}{24} \left[- \frac{31}{36} \frac{p'^4}{p^3} + \frac{149}{72} \frac{p'^2 p''}{p^2} - \right. \\
& \left. \left. - \frac{103}{144} \frac{p''^2}{p} + \frac{77}{288} p^{(4)} \right] h^4 + o(h^5) \right\},
\end{aligned}$$

$$\begin{aligned} \frac{\hat{D}(\frac{1}{n}, \frac{2}{n})}{\hat{D}(0, \frac{2}{n})} &= \frac{1}{2} - \frac{1}{4} \frac{p'}{p} h + \frac{1}{4} \left[\frac{p'^2}{p^2} - \frac{p''}{p} \right] h^2 + \\ &+ \left[\frac{-7}{24} \frac{p'^3}{p^3} + \frac{11}{24} \frac{p'p''}{p^2} - \frac{7}{48} \frac{p'''}{p} \right] h^3 + o(h^4), \\ f(\frac{1}{n}, y(\frac{1}{n})) &= (py')' + h(py')'' + \frac{h^2}{2} (py')^{(3)} + \\ &+ \frac{h^3}{6} (py')^{(4)} + o(h^4), \end{aligned}$$

where all the functions and their derivatives have been evaluated at the point $x = 0$. We get

$$\begin{aligned} &\frac{1}{\hat{D}(0, \frac{2}{n})} [y(0) - y(\frac{2}{n})] - \frac{h}{3} \left[f(0, y(0)) + 4 \frac{\hat{D}(\frac{1}{n}, \frac{2}{n})}{\hat{D}(0, \frac{2}{n})} f(\frac{1}{n}, y(\frac{1}{n})) \right] - \\ &- \frac{a_{11}p(0)}{a_{12}} y(0) + \frac{cp(0)}{a_{12}} = \\ &= h^4 F(y', y'', y^{(3)}, y^{(4)}, y^{(5)}, p, p', p'', p^{(3)}, p^{(4)}) + o(h^5), \end{aligned}$$

where F is a polynomial of the variables $y', \dots, y^{(5)}, p, \dots, p^{(4)}$.

Hence we have

$$\hat{\tau}_0(h) = h^4 F - \frac{\hat{D}(\frac{1}{n}, \frac{2}{n})}{\hat{D}(0, \frac{2}{n})} \hat{\tau}_1(h) + o(h^5) = o(h^4).$$

By symmetry we also have $\hat{\tau}_n(h) = o(h^4)$.

Let $Y (y_0, y_1, \dots, y_n)^T$ be the solution of (5.38) with $\hat{\tau}(h) = 0$.

As above, Theorem 5.9 applies and we have

$$(5.39) \quad \max_{i=0, \dots, n} \left| y(\frac{1}{n}) - y_i \right| = o(h^4),$$

if $0 \leq f'_y(x, y) \leq e$ on $[0, 1] \times R$. It is also clear that Theorem 5.10

applies. Further more these results also hold when $a_{12} = 0$ or $b_{22} = 0$.

6. PEANO'S KERNEL THEOREM

In this chapter we are going to consider the following class of linear functionals

$$(6.1) \quad u(f) = \int_a^b [a_0(t)f(t) + a_1(t)f'(t) + \dots + a_n(t)f^{(n)}(t)] dt + \\ + \sum_{i=0}^{k_0} b_{i,0}f(x_{i,0}) + \sum_{i=0}^{k_1} b_{i,1}f'(x_{i,1}) + \dots + \sum_{i=0}^{k_n} b_{i,n}f^{(n)}(x_{i,n}),$$

where $f \in C^{n+1}$, and the functions $a_j(x)$ are assumed to be continuous over the interval $[a,b]$ and the points $x_{i,j}$ to lie in $[a,b]$.

We will derive Peano's kernel theorem for u in the sense of distributions.

Put $a_j(t) = 0$ for $t \notin [a,b]$, $j = 0, \dots, n$, then $\text{supp } a_j \subset [a,b]$ for $j = 0, 1, \dots, n$. u is a linear and continuous form on $C_0^n(\mathbb{R})$ and we have

$$(6.2) \quad u = \sum_{j=0}^n \left(-\frac{d}{dt}\right)^j [a_j(t) + \sum_{i=0}^{k_j} b_{i,j} \delta_{x_{i,j}}].$$

Further more since $\text{supp } a_j \subset [a,b]$ we have $\text{supp } u \subset [a,b]$ and $u \in \mathcal{D}'(\mathbb{R})$.

We shall now introduce two concepts from the theory of distributions, namely the primitives of distributions and convolution of distributions.

We only give a brief outline of these concepts here. For more information see Bremermann [9], Friedman [12], Hörmander [15], and Zemanian [28].

Let $T \in \mathcal{D}'(\mathbb{R})$ then a distribution S such that

$$\langle S', \varphi \rangle = \langle T, \varphi \rangle \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R})$$

is called a primitive distribution of T . A distribution Q such that

$$\langle Q^{(m)}, \varphi \rangle = \langle T, \varphi \rangle \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R})$$

is called a primitive distribution of T of order m . Any distribution in $\mathcal{D}'(R)$ has primitive distributions of any order.

The primitive distribution of m th order is unique up to an additive polynomials of degree $\leq m-1$. If $T \in \mathcal{D}'^m(R)$ and S is a primitive distribution of T of first order, then $S \in \mathcal{D}'^{m-1}(R)$.

If u and v are in $C(R)$ and either one has compact support, then the convolution $u \ast v$ is the continuous function defined by

$$(u \ast v)(x) = \int u(x-t)v(t)dt, \quad x \in R.$$

If we take $x-t$ as a new integration variable we obtain

$$u \ast v = v \ast u.$$

The convolution between a distribution $u \in \mathcal{D}'(R)$ and a function

$\varphi \in C_0^\infty(R)$ is defined by

$$(u \ast \varphi)(x) = u(\varphi(x - \cdot)),$$

where the right hand side denotes u acting on the function

$t \rightarrow \varphi(x-t)$. The convolution $u_1 \ast u_2$ of two distributions u_1 and u_2

one of which has compact support is defined to be the unique

distribution u such that

$$u_1 \ast (u_2 \ast \varphi) = u \ast \varphi, \quad \varphi \in C_0^\infty(R),$$

is valid.

The convolution is associative, that is, $u_1 \ast (u_2 \ast u_3) = (u_1 \ast u_2) \ast u_3$

if only one of the distributions u_j does not have compact support.

The convolution is commutative, that is, $u_1 \ast u_2 = u_2 \ast u_1$ if one of the distributions u_1, u_2 has compact support.

We have

$$\delta \ast \varphi(x) = \varphi(x),$$

$$\delta_{x_1} \ast \varphi(x) = \varphi(x - x_1),$$

for $\varphi \in C^0(R)$, and

$$\delta^{(j)} * \varphi(x) = \delta * \varphi^{(j)}(x) = \varphi^{(j)}(x),$$

$$\delta_{x_1}^{(j)} * \varphi(x) = \varphi^{(j)}(x - x_1),$$

for $\varphi \in C^j(\mathbb{R})$.

Differentiation can be interpreted as a convolution, for let D be the differential operator, then we have

$$(D^k u) * \varphi = u * (D^k \varphi) = u * (\delta * (D^k \varphi)) = u * \delta^{(k)} * \varphi.$$

Thus

$$D^k u = \delta^{(k)} * u.$$

If u_1, u_2 are two distributions, one of which has compact support, it follows that

$$D^k(u_1 * u_2) = (D^k u_1) * u_2 = u_1 * D^k u_2.$$

In fact, if the differentiations are rewritten as convolutions with $\delta^{(k)}$ this follows from the associativity and commutativity of convolution.

Consider the distribution u defined by (6.2). We will try to find a primitive distribution of u of order $n+1$ with respect to $-d/dx$. Hence we want to solve the differential equation

$$\left(-\frac{d}{dt}\right)^{n+1} K = u$$

for K . Consider the function

$$(6.3) \quad H_k(t) = \begin{cases} (-t)^k/k! , & t \leq 0, \\ 0, & t > 0. \end{cases}$$

Then $H_k \in C^{k-1}(\mathbb{R})$ and we have, in the sense of distributions,

$$(6.4) \quad \left(-\frac{d}{dt}\right)^{n+1} H_n(t-x) = \delta_x.$$

Hence $H_n(t-x)$ is a primitive distribution of δ_x of order $n+1$

with respect to $-d/dt$. $H_n(t-x)$ is also called a fundamental solution

for $(-\frac{d}{dt})^{n+1}$. Now consider the convolution $u \ast H_n$, where u has compact support. We have ($D = d/dt$)

$$(-D)^{n+1}(u \ast H_n) = u \ast (-D)^{n+1}H_n = u \ast \delta = u.$$

Hence $u \ast H_n$ is a primitive distribution of u of order $n+1$ with respect to $-d/dt$. Thus put

$$\begin{aligned} K(t) &= u \ast H_n(t) = \\ &= \left\{ \sum_{j=0}^n (-D)^j [a_j + \sum_{i=0}^{k_j} b_{i,j} \delta_{x_{i,j}}] \right\} \ast H_n(t) = \\ &= \left\{ \sum_{j=0}^n (-1)^j [a_j \ast \delta^{(j)} + \sum_{i=0}^{k_j} b_{i,j} \delta_{x_{i,j}}^{(j)}] \right\} \ast H_n(t). \end{aligned}$$

For $j \leq n$ we have

$$(-1)^j \delta_x^{(j)} \ast H_n(t) = (-\frac{d}{dt})^j H_n(t-x) = H_{n-j}(t-x)$$

in the sense of distributions. Put

$$t_+^k = \begin{cases} t^k, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

then $H_k(t-x) = (x-t)_+^k/k!$ and

$$(6.5) \quad K(t) = \sum_{j=0}^n \left\{ \int_a^b \frac{(x-t)_+^{n-j}}{(n-j)!} a_j(x) dx + \sum_{i=0}^{k_j} b_{i,j} \frac{(x_{i,j}-t)_+^{n-j}}{(n-j)!} \right\}.$$

K is a primitive distribution of u of order $n+1$ with respect to $-d/dt$. We also have $\text{supp } K \subset [-\infty, b]$. Thus $K(t) + P_n(t)$, where P_n is a polynomial of degree $\leq n$, is also a primitive distribution of u of order $n+1$. But we will assume that $\text{supp}(K + P_n) \subset [-\infty, b]$ then we must have $P_n = 0$.

We now have

$$(6.6) \quad u(\varphi) = \left\langle \left(-\frac{d}{dt}\right)^{n+1} u * H_n(t), \varphi \right\rangle = \left\langle u * H_n(t), \varphi^{(n+1)} \right\rangle = \\ = \left\langle K(t), \varphi^{(n+1)} \right\rangle = \int_{-\infty}^b K(t) \varphi^{(n+1)}(t) dt, \quad \varphi \in C_0^{n+1}(R),$$

since K is continuous.

Theorem 6.1

K has compact support if and only if

$$u_x \left(\frac{(x-t)^n}{n!} \right) = 0 \quad \text{for all } t.$$

Proof: If K has compact support then $\left(\frac{d}{dt}\right)^j K(t)$, $j = 0, \dots, n+1$, also have compact support. Hence

$$\left(\frac{d}{dt}\right)^j K(t) \in \mathcal{D}'^{(n+1)}(R), \quad j = 0, \dots, n+1.$$

Since $(x-t)^n/n! \in C^{n+1}(R)$ we have

$$u_x \left(\frac{(x-t)^n}{n!} \right) = \left\langle \left(-\frac{d}{dx}\right)^{n+1} u * H_n, \frac{(x-t)^n}{n!} \right\rangle = \\ = \left\langle K, \left(\frac{d}{dx}\right)^{n+1} \frac{(x-t)^n}{n!} \right\rangle = \left\langle K, 0 \right\rangle = 0.$$

If $u_x \left(\frac{(x-t)^n}{n!} \right) = 0$ for all t , we have

$$0 = u_x \left(\frac{(x-t)^n}{n!} \right) = \sum_{j=0}^n \left\{ \int_a^b \frac{(x-t)^{n-j}}{(n-j)!} a_j(x) dx + \sum_{i=0}^k \frac{(x_{i,j} - t)^{n-j}}{(n-j)!} \right\}.$$

Hence $u_x \left(\frac{(x-t)^n}{n!} \right) = K(t) = 0$ for $t \leq a$ and $\text{supp } K \subset [a, b]$.

Remark 1.

Let $\mathcal{P}_n$ be the class of all polynomials of degree $\leq n$. Then the

hypothesis $u_x \left(\frac{(x-t)^n}{n!} \right) = 0$ for all t is equivalent to $\mathcal{P}_n \subset \ker u$.

Remark 2.

In Davis [10] p. 70 $K(t)$ is defined by

$$(6.7) \quad K(t) = \frac{1}{n!} u_x((x-t)_+^n),$$

which does not make sense since $(x-t)_+^n$ is in $C^{n-1}(R)$ but not in $C^n(R)$. However if we interpret (6.7) in the Sobolev space W^n (see chapter 7) it does make sense.

Theorem 6.2 (Peano's kernel theorem)

Let $u(p) = 0$ for all $p \in \mathcal{P}_n$. Then

$$(6.8) \quad u(\varphi) = \int_a^b K(t) \varphi^{(n+1)}(t) dt, \quad \varphi \in C^{n+1}(R),$$

where

$$(6.9) \quad K(t) = u \# H_n(t) = \\ = \sum_{j=0}^n \left\{ \int_a^b \frac{(x-t)_+^{n-j}}{(n-j)!} a_j(x) dx + \sum_{i=0}^{k_j} b_{i,j} \frac{(x_{i,j}-t)_+^{n-j}}{(n-j)!} \right\}.$$

Proof: The theorem follows immediately from Theorem 6.1 and (6.6) since $\text{supp } K \subset [a, b]$.

If

$$(6.10) \quad \hat{u} = \sum_{j=0}^{n-1} \left(-\frac{d}{dt} \right)^j \left[a_j(t) + \sum_{i=0}^{k_j} b_{i,j} \delta_{x_{i,j}} \right]$$

and $\hat{u}(p) = 0$ for all $p \in \mathcal{P}_n$, then, by the definition of convolution between a distribution and a testfunction, the corresponding $\hat{K}$ is given by

$$(6.11) \quad \hat{K}(t) = \frac{1}{n!} u_x((x-t)_+^n)$$

and we have

$$(6.12) \quad \hat{u}(\varphi) = \int_a^b \hat{K}(t) \varphi^{(n+1)}(t) dt, \quad \varphi \in C^{n+1}(R).$$

Let

$$(6.13) \quad u_m = \sum_{j=0}^m \left(-\frac{d}{dt}\right)^j [a_j(t) + \sum_{i=0}^{k_j} b_{i,j} \delta_{x_{i,j}}],$$

where $m < n$. Put

$$(6.14) \quad K_r(t) = u_m * H_r(t), \quad m \leq r \leq n,$$

then $(K_n(t) = K(t))$ and we have the following theorem.

Theorem 6.3

Let $u_m(p) = 0$ for all $p \in \mathcal{P}_n$. Then

$$(6.15) \quad u_m(\varphi) = \int_a^b K_r(t) \varphi^{(r+1)}(t) dt, \quad \varphi \in C^{r+1}(R),$$

for $r = m, m+1, \dots, n$, where

$$(6.16) \quad \begin{cases} K_m(t) = u_m * H_m(t), \\ K_r(t) = \frac{1}{r!} u_{m,x}((x-t)_+^r), \quad r > m. \end{cases}$$

Proof: The proof is exactly the same as for Theorem 6.2 except for (6.16) when $r > m$. But since

$$(x-t)_+^r \in C^m(R) \quad \text{if } r > m,$$

we have

$$K_r(t) = u_m * H_r(t) = u_{m,x}(H_r(t-x)) = u_{m,x}\left(\frac{(x-t)_+^r}{r!}\right),$$

where $u_{m,x}$ denotes u_m acting on the function $x \rightarrow (x-t)_+^r/r!$.

Example 6.1

If $y \in C^n$ then, by Peano's kernel theorem, the n th divided difference of y can be written as

$$y(x_0, x_1, \dots, x_n) = \frac{1}{n!} \sum_{x_0}^{x_n} M_n(x; x_0, \dots, x_n) y^{(n)}(x) dx,$$

where

$$\begin{cases} M_n(x; x_0, \dots, x_n) = \sum_{v=0}^n n(x_v - x)_+^{n-1} / w'(x), \\ w(x) = (x - x_0) \dots (x - x_n). \end{cases}$$

For $n = 2$ we have

$$\begin{aligned} y(x_0, x_1, x_2) &= \frac{\frac{y(x_2) - y(x_1)}{x_2 - x_1} - \frac{y(x_1) - y(x_0)}{x_1 - x_0}}{x_2 - x_0} = \\ &= \left\{ \frac{-y(x_0)}{x_0 - x_1} + \left[\frac{1}{x_0 - x_1} + \frac{1}{x_1 - x_2} \right] y(x_1) + \frac{-y(x_2)}{x_1 - x_2} \right\} / (x_2 - x_0), \end{aligned}$$

and

$$M_2(x; x_0, x_1, x_2) = 2 \begin{cases} \frac{x_0 - x}{(x_0 - x_1)(x_2 - x_0)}, & x_0 \leq x \leq x_1, \\ \frac{x_1 - x_2}{(x_2 - x_0)(x_1 - x_2)}, & x_1 \leq x \leq x_2. \end{cases}$$

Hence if $y'' = f$ we have

$$\begin{aligned} &\frac{-y(x_0)}{x_0 - x_1} + \left[\frac{1}{x_0 - x_1} + \frac{1}{x_1 - x_2} \right] y(x_1) + \frac{-y(x_2)}{x_1 - x_2} = \\ &= \int_{x_0}^{x_1} \frac{x_0 - x}{x_0 - x_1} f(x) dx + \int_{x_1}^{x_2} \frac{x - x_2}{x_1 - x_2} f(x) dx, \end{aligned}$$

which is the determinant formulation of $y'' = f$ with $a(x) = x$,

$$b(x) = x - 1.$$

Peano's kernel theorem, formulated in Banach spaces, is a special case of Sard's quotient theorem:

Theorem 6.4

Suppose that $X, \tilde{X}, Y$ are Banach spaces, that $U: X \rightarrow \tilde{X}$ is a surjective linear continuous operator, and that $P: X \rightarrow Y$ is a linear continuous operator. If

$$(6.17) \quad Px = 0 \text{ whenever } Ux = 0, \quad x \in X$$

then there exists a linear continuous operator $Q: \tilde{X} \rightarrow Y$ such that

$$Px = QUx, \quad x \in X.$$

Proof: See Sard [20] p. 311 or Holmes [13] p. 176.

We may think of Q as the quotient P/U but the theorem does not say anything about the construction of Q . In the proof of Peano's kernel theorem we used the space $C^{n+1}(R)$ on which we put the topology defined by: a sequence $\varphi_j \in C^{n+1}(R)$ is tending to 0 in the sense that $\varphi_j^{(k)} \rightarrow 0$ uniformly on any compact subset of R for every $k \leq n+1$. Then $C^{n+1}(R)$ is a Frechet space. The linear and continuous forms on this space are in $\mathcal{G}^{(n+1)}(R)$. We will continue to work with these spaces.

In Peano's kernel theorem $(d/dt)^{n+1}$ corresponds to the operator U in Sard's quotient theorem. We will now consider the differential operator $U = L = \frac{d}{dt}(p \frac{d}{dt}) + q$, where $p \in C^1[0,1]$, $q \in C[0,1]$, and $p > 0$ in $[0,1]$. Let $J = (j_1, j_2)$ be an open interval containing $[0,1]$. Then we can continue p and $1/p$ to C^1 -functions on J and also q to a continuous function on J . Consider the linear form

$$(6.18) \quad u = d_1 \delta_{x_1} + d_2 \delta_{x_2} + d_3 \delta_{x_3}, \quad 0 \leq x_1 < x_2 < x_3 \leq 1.$$

We have $u \in \mathcal{G}^{(2)}(J)$.

As for Peano's kernel theorem we want to solve the differential equation

$$(6.19) \quad LK = u$$

for K . We define a fundamental solution $E(x, t)$, with a pole at t , for L by

$$L_x E = \delta_t, \quad x, t \in J.$$

There exists two fundamental solutions $E_+(x, t)$ and $E_-(x, t)$ which are uniquely defined by $E_+(x, t) = 0$ for $x < t$ and $E_-(x, t) = 0$ for $x > t$. To solve (6.19) we will consider $E_-(x, t)$ and a fundamental system $\{y_1, y_2\}$ of solutions of the homogeneous equation $Ly = 0$ on J .

Put

$$E_-(x, t) = \begin{cases} \alpha_1 y_1(x) + \alpha_2 y_2(x), & x \leq t, \\ 0, & x > t. \end{cases}$$

We will choose the constants α_1 and α_2 such that $L_x E_- = \delta_t$, or

$$\int_{J_1}^t E_-(x, t) L\varphi(x) dx = \varphi(t), \quad \varphi \in C_0^2(J).$$

Integration by parts give

$$\begin{cases} \alpha_1 y_1(t) + \alpha_2 y_2(t) = 0, \\ \alpha_1 y_1'(t) + \alpha_2 y_2'(t) = -1/p(t). \end{cases}$$

Since the Wronskian

$$W(y_1, y_2, t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0$$

we have

$$(6.20) \quad E_-(x, t) = \begin{cases} - \begin{vmatrix} y_2(x) & y_1(x) \\ y_2(t) & y_1(t) \end{vmatrix} / [p(t)W(y_1, y_2, t)], & x \leq t, \\ 0, & x > t. \end{cases}$$

Hence

$$(6.21) \quad K(t) = d_1 E_-(t, x_1) + d_2 E_-(t, x_2) + d_3 E_-(t, x_3)$$

is a solution of (6.19) and $\text{supp } K \subset [j_1, x_3]$. We now have

$$(6.22) \quad u(\varphi) = \langle LK, \varphi \rangle = \langle K, L\varphi \rangle = \\ = \int_{j_1}^{x_3} K(t) L\varphi(t) dt, \quad \varphi \in C_0^2(J),$$

since K is continuous.

Theorem 6.5

K has compact support if and only if $u(y) = 0$ for all y such that $Ly = 0$.

Proof: If K has compact support in J then K' and K'' also have compact support. Hence

$$LK \in \mathcal{G}'^2(J).$$

Assume that $Ly = 0$, then $y \in C^2(J)$ and we have

$$u(y) = \langle LK, y \rangle = \langle K, Ly \rangle = \langle K, 0 \rangle = 0.$$

If $u(y) = 0$ for all y such that $Ly = 0$, then

$$L_x \frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x) & y_1(x) \end{vmatrix}}{p(x)W(y_1, y_2, x)} = 0$$

since $p(x)W(y_1, y_2, x)$ is constant, and we have

$$0 = u_x \left(\frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x) & y_1(x) \end{vmatrix}}{p(x)W(y_1, y_2, x)} \right) = -d_1 \frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x_1) & y_1(x_1) \end{vmatrix}}{p(x_1)W(y_1, y_2, x_1)} -$$

$$-d_2 \frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x_2) & y_1(x_2) \end{vmatrix}}{p(x_2)W(y_1, y_2, x_2)} - d_3 \frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x_3) & y_1(x_3) \end{vmatrix}}{p(x_3)W(y_1, y_2, x_3)}.$$

Hence

$$0 = u_x \left(\frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x) & y_1(x) \end{vmatrix}}{p(x)W(y_1, y_2, x)} \right) = K(t), \quad \text{for } t \leq x_1.$$

and $\text{supp } K \subset [x_1, x_3]$.

Theorem 6.6

Let $u(y) = 0$ for all y such that $Ly = 0$. Then

$$(6.23) \quad u(\varphi) = \int_{x_1}^{x_3} K(t)L\varphi(t)dt, \quad \varphi \in C^2(J),$$

where

$$(6.24) \quad K(t) = d_1 E_-(t, x_1) + d_2 E_-(t, x_2) + d_3 E_-(t, x_3).$$

Proof: The theorem follows immediately from Theorem 6.5 and (6.22).

Since

$$u(y) = 0 \quad \text{when } Ly = 0$$

we must have some conditions on the constants d_1, d_2 , and d_3 .

Let y_1, y_2 be two linearly independent solutions of $Ly = 0$. Then we have

$$\begin{cases} d_1 y_1(x_1) + d_2 y_1(x_2) + d_3 y_1(x_3) = 0, \\ d_1 y_2(x_1) + d_2 y_2(x_2) + d_3 y_2(x_3) = 0. \end{cases}$$

We assume that

$$\begin{vmatrix} y_1(x_1) & y_1(x_2) \\ y_2(x_1) & y_2(x_2) \end{vmatrix} \begin{vmatrix} y_1(x_2) & y_1(x_3) \\ y_2(x_2) & y_2(x_3) \end{vmatrix} \neq 0.$$

Then we have

$$\begin{cases} d_1 = -AD(x_2, x_3), \\ d_2 = AD(x_1, x_3), \\ d_3 = -AD(x_1, x_2), \end{cases}$$

where

$$D(s, t) = \begin{vmatrix} y_2(s) & y_1(s) \\ y_2(t) & y_1(t) \end{vmatrix},$$

and A is an arbitrary constant. We also have

$$K(t) = A [-D(x_2, x_3)E_-(t, x_1) + D(x_1, x_3)E_-(t, x_2) - D(x_1, x_2)E_-(t, x_3)] =$$

$$= A \begin{cases} \frac{D(x_2, x_3)D(x_1, t)}{C_{pW}}, & x_1 \leq t \leq x_2, \\ \frac{D(x_1, x_2)D(t, x_3)}{C_{pW}}, & x_2 \leq t \leq x_3, \end{cases}$$

where $C_{pW} = p(x)W(y_1, y_2, x)$ is constant.

Hence equation (6.23) is equivalent to the determinant formulation (3.5).

We will also show that (3.7) and (3.9) can be obtained from a kernel theorem. Consider the linear form

$$u = d_1 \delta_{x_1} + e_1 \delta'_{x_1} + d_2 \delta_{x_2}, \quad 0 \leq x_1 < x_2 \leq 1.$$

Then $u \in \mathcal{G}'^2(J)$. The differential equation

$$(6.25) \quad LK = u$$

has a solution

$$(6.26) \quad K(t) = d_1 E_-(t, x_1) + e_1 F_-(t, x_1) + d_2 E_-(t, x_2)$$

in the sense of distribution, where

$$(6.27) \quad F_-(t, x) = \begin{cases} \frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2'(x) & y_1'(x) \end{vmatrix}}{[p(x)W(y_1, y_2, x)]}, & x \geq t, \\ 0, & x < t, \end{cases}$$

since $\delta'_t(\varphi) = -\varphi'(t)$. Hence K is a solution of (6.25) and $\text{supp } K \subset [j_1, x_2]$. We now have

$$(6.28) \quad u(\varphi) = \langle LK, \varphi \rangle = \langle K, L\varphi \rangle = \int_{j_1}^{x_2} K(t)L\varphi(t)dt, \quad \varphi \in C_0^2(J),$$

since K is piecewise continuous.

Theorem 6.7

K has compact support if and only if $u(y) = 0$ for all y such that $Ly = 0$.

Proof: If K has compact support in J then K' and K'' also have compact support in J . Hence

$$LK \in \mathcal{G}^{2,2}(J).$$

Assume $Ly = 0$, then $y \in C^2(J)$ and we have

$$u(y) = \langle LK, y \rangle = \langle K, Ly \rangle = \langle K, 0 \rangle = 0.$$

If $u(y) = 0$ for all y such that $Ly = 0$, then

$$L_x \frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x) & y_1(x) \end{vmatrix}}{p(x)W(y_1, y_2, x)} = 0$$

since $p(x)W(y_1, y_2, x)$ is constant, and we have

$$\begin{aligned} 0 &= u_x \left(\frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x) & y_1(x) \end{vmatrix}}{p(x)W(y_1, y_2, x)} \right) = -d_1 \frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x_1) & y_1(x_1) \end{vmatrix}}{p(x_1)W(x_1, y_1, y_2)} + \\ &+ e_1 \frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2'(x_1) & y_1'(x_1) \end{vmatrix}}{p(x_1)W(y_1, y_2, x_1)} - d_2 \frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x_2) & y_1(x_2) \end{vmatrix}}{p(x_2)W(y_1, y_2, x_2)}. \end{aligned}$$

Hence

$$0 = u_x \left(\frac{\begin{vmatrix} y_2(t) & y_1(t) \\ y_2(x) & y_1(x) \end{vmatrix}}{p(x)W(y_1, y_2, x)} \right) = K(t), \quad \text{for } t < x_1,$$

and $\text{supp } K \subset [x_1, x_2]$.

Theorem 6.8

Let $u(y) = 0$ for all y such that $Ly = 0$. Then

$$(6.29) \quad u(\varphi) = \int_{x_1}^{x_2} K(t)L\varphi(t)dt, \quad \varphi \in C^2(J),$$

where

$$(6.30) \quad K(t) = d_1 E_-(t, x_1) + e_1 F_-(t, x_1) + d_2 E_-(t, x_2).$$

Proof: The theorem follows immediately from Theorem 6.7 and (6.28).

Since

$$u(y) = 0 \quad \text{when } Ly = 0$$

we must have some conditions on the constants d_1 , e_1 , and d_2 .

Let y_1, y_2 be two linearly independent solutions of $Ly = 0$.

Then we have

$$\begin{cases} d_1 y_1(x_1) - e_1 y_1'(x_1) + d_2 y_1(x_2) = 0, \\ d_1 y_2(x_1) - e_1 y_2'(x_1) + d_2 y_2(x_2) = 0. \end{cases}$$

since

$$-y_1(x_1)y_2'(x_1) + y_2(x_1)y_1'(x_1) = -W(y_1, y_2, x) \neq 0$$

we can solve this system for d_1 and e_1 , we get

$$\begin{cases} d_1 = -A \begin{vmatrix} y_2'(x_1) & y_1'(x_1) \\ y_2(x_2) & y_1(x_2) \end{vmatrix}, \\ e_1 = -AD(x_1, x_2) \\ d_2 = AW(y_1, y_2, x_1), \end{cases}$$

where A is an arbitrary constant. Hence if $Ly = 0$ we have

$$\begin{aligned} & - \begin{vmatrix} y_2'(x_1) & y_1'(x_1) \\ y_2(x_2) & y_1(x_2) \end{vmatrix} y(x_1) + D(x_1, x_2) y'(x_1) + W(y_1, y_2, x_1) y(x_2) = \\ & = - \begin{vmatrix} y_2(x_1) & y_1(x_1) & y(x_1) \\ y_2'(x_1) & y_1'(x_1) & y'(x_1) \\ y_2(x_2) & y_1(x_2) & y(x_2) \end{vmatrix} = 0. \end{aligned}$$

We also have

$$\begin{aligned} K(t) = A \left\{ - \begin{vmatrix} y_2'(x_1) & y_1'(x_1) \\ y_2(x_2) & y_1(x_2) \end{vmatrix} E_-(t, x_1) - D(x_1, x_2) F_-(t, x_1) + \right. \\ \left. + W(y_1, y_2, x_1) E_-(t, x_2) \right\}. \end{aligned}$$

Hence for $x_1 \leq t \leq x_2$ we have

$$K(t) = AW(y_1, y_2, x_1) \frac{-D(t, x_2)}{p(x_1)W(y_1, y_2, x_1)} = A \frac{-D(t, x_2)}{p(x_1)},$$

and for $t < x_1$ we have

$$K(t) = -\frac{A}{C_{PW}} \begin{vmatrix} y_2(t) & y_1(t) & \begin{vmatrix} y_2(t) & y_1(t) \\ y_2'(x_1) & y_1'(x_1) \end{vmatrix} \\ y_2(x_1) & y_1(x_1) & \begin{vmatrix} y_2(x_1) & y_1(x_1) \\ y_2'(x_1) & y_1'(x_1) \end{vmatrix} \\ y_2(x_2) & y_1(x_2) & \begin{vmatrix} y_2(x_2) & y_1(x_2) \\ y_2'(x_1) & y_1'(x_1) \end{vmatrix} \end{vmatrix} = 0.$$

Hence (6.29) can be written as

$$\begin{aligned}
 & -A \begin{vmatrix} y_2'(x_1) & y_1'(x_1) \\ y_2(x_2) & y_1(x_2) \end{vmatrix} y(x_1) + AD(x_1, x_2) y'(x_1) + AW(y_1, y_2, x_1) y(x_2) = \\
 & = -A \int_{x_1}^{x_2} \frac{D(t, x_2)}{p(x_1)} Ly(t) dt,
 \end{aligned}$$

or (if $A \neq 0$)

$$(6.31) \quad p(x_1) \begin{vmatrix} y_2(x_1) & y_1(x_1) & y(x_1) \\ y_2'(x_1) & y_1'(x_1) & y'(x_1) \\ y_2(x_2) & y_1(x_2) & y(x_2) \end{vmatrix} = \int_{x_1}^{x_2} D(t, x_2) Ly(t) dt.$$

Assume that

$$\begin{cases} B_1 y = a_{11} y(x_1) - a_{12} y'(x_1) = c, & a_{12} > 0, \\ B_1 y_1 = 0, \end{cases}$$

then we have

$$\begin{aligned}
 & p(x_1) \begin{vmatrix} y_2(x_1) & y_1(x_1) & y(x_1) \\ a_{12} y_2'(x_1) & a_{12} y_1'(x_1) & a_{11} y(x_1) - c \\ y_2(x_2) & y_1(x_2) & y(x_2) \end{vmatrix} = \\
 & = a_{12} \int_{x_1}^{x_2} D(t, x_2) Ly(t) dt.
 \end{aligned}$$

Expansion of the determinant after the third column gives

$$\begin{aligned}
 & -B_1(y_2)y_1(x_2)y(x_1) + cD(x_1, x_2) + B_1(y_2)y_1(x_1)y(x_2) = \\
 & = \frac{a_{12}}{p(x_1)} \int_{x_1}^{x_2} D(t, x_1) Ly(t) dt,
 \end{aligned}$$

or

$$\begin{aligned}
 (6.32) \quad & \frac{y_1(x_2)}{y_1(x_1)D(x_1, x_2)} y(x_1) - \frac{y(x_2)}{D(x_1, x_2)} = \\
 & = \frac{-a_{12}}{p(x_1)B_1(y_2)y_1(x_1)} \int_{x_1}^{x_2} \frac{D(t, x_2)}{D(x_1, x_2)} Ly(t) dt + \frac{c}{y_1(x_1)B_1(y_2)}.
 \end{aligned}$$

Since $1/B_1(y_2) = -p(x_1)y_1(x_1)/a_{12}$ equation (6.32) is equivalent to (3.7).

We can also give a kernel theorem for equation (3.9), but since the computations are exactly the same as for (3.7) we omit it.

7. GALERKIN'S METHOD

Consider the differential equation

$$(7.1) \quad Ly = (py')' + qy = f(x), \quad 0 < x < 1,$$

where $p \in C^1[0,1]$, $p > 0$ on $[0,1]$ and $q, f \in C[0,1]$.

We will consider the weak forms of the differential equation (7.1).

There are several, but they all share the following basic principle:

The equation is multiplied by test functions $v(x)$ and integrated over the interval $[0,1]$, to yield

$$(7.2) \quad \int_0^1 (Ly)v \, dx = \int_0^1 f v \, dx.$$

This is to hold for each function v in some test space V . The weak formulation can be used to share the regularity requirements between the solution y and the test function v .

Define the spaces W^m and W_0^m by

$$W^m = W^{m,2}[0,1] = \{ u \in C^{m-1}[0,1] \mid u^{(m-1)} \text{ is absolutely continuous, } u^{(m)} \in L^2[0,1] \},$$

$$W_0^m = W_0^{m,2}[0,1] = \{ u \in W^m \mid u^{(k)}(0) = u^{(k)}(1) = 0, \, k=0, \dots, m-1 \},$$

where m is a nonnegative integer. Equipped with the norm

$$\|u\|_m = \left\{ \sum_{j=0}^m \int_0^1 (u^{(j)}(t))^2 dt \right\}^{\frac{1}{2}},$$

these are called Sobolev spaces over $[0,1]$. Clearly $W^0[0,1] = L^2[0,1]$.

If $V = W^0$ equation (7.2) is said to hold in the strong sense.

With $f \in L^2[0,1]$ the solution y will be sought in W^2 . If $V = W^1$

a formal integration by parts gives the equation

$$(7.3) \quad \int_0^1 [-p(t)y'(t)v'(t) + q(t)y(t)v(t)] dt + p(1)y'(1)v(1) - \\ - p(0)y'(0)v(0) = \int_0^1 f(t)v(t) dt, \quad v \in W^1,$$

and if $v = W_0^1$ we get

$$(7.4) \quad \int_0^1 [-p(t)y'(t)v'(t) + q(t)y(t)v(t)] dt = \int_0^1 f(t)v(t) dt, \quad v \in W_0^1.$$

In these manipulations, the boundary conditions must play their part.

With equation (7.3) the full set of boundary conditions ($a_{12}b_{22} > 0$)

$$(7.5) \quad \begin{cases} B_1 y = a_{11}y(0) - a_{12}y'(0) = c, \\ B_2 y = b_{21}y(1) + b_{22}y'(1) = d, \end{cases}$$

is imposed on y and the solution must lie in W^2 . For $y \in W^1$ the pointwise definition of $y'(0)$ and $y'(1)$ have no meaning since $y' \in L^2[0,1]$. In equation (7.4) the solution is sought in W^1 and the only boundary conditions which make sense are

$$\begin{cases} B_1 y = a_{11}y(0) = c, \\ B_2 y = b_{21}y(1) = d. \end{cases}$$

Galerkin's method is the obvious discretization of the weak forms

(7.3) and (7.4). This method involves two families of functions; a subspace S_n of the solution space and a subspace V_n of the test space V .

Consider the differential equation (7.1) with boundary conditions given by (7.5). We assume that Green's function for this problem is

$$(7.6) \quad G(x,t) = \begin{cases} b(x)a(t), & x \geq t, \\ a(x)b(t), & x \leq t, \end{cases}$$

where $a, b \in C^2[0,1]$. The weak form of this problem is given by (7.3)

and the solution space is W^2 . Let $S_n = W^2$, and

$$V_n = \{v_n \mid v_n = c_0 a_{12} \Delta_0 + \sum_{j=1}^{n-1} c_j \Delta_j + c_n b_{22} \Delta_n\},$$

where Δ_j , $j = 0, \dots, n$, are defined by (3.6), (3.8), and (3.10).

Then the Galerkin solution y_n is the element in W^2 which satisfies

$$(7.7) \quad \int_0^1 [-p(t)y_n'(t)v_n'(t) + q(t)y_n(t)v_n(t)]dt + p(1)y_n'(1)v_n(1) - \\ - p(0)y_n'(0)v_n(0) = \int_0^1 f(t)v_n(t)dt,$$

and the boundary condition (7.5) for all v_n in V_n . Put $v_n := \Delta_i$,

$i = 1, \dots, n-1$, in (7.7), then by integration by parts, we get

$$(7.8) \quad p(x_{i-1})\Delta_i'(x_{i-1}+0)y_n(x_{i-1}) + p(x_i)[\Delta_i'(x_i+0) - \Delta_i'(x_i-0)]y_n(x_i) - \\ - p(x_{i+1})\Delta_i'(x_{i+1}-0)y_n(x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} f(t)\Delta_i(t)dt,$$

for $i = 1, \dots, n-1$. By (3.29) and (3.27) we see that equation

(7.8) is exactly equation (3.5) if we replace y_n by y . If

$a_{12} > 0$ put

$$y'(0) = \frac{a_{11}y(0) - c}{a_{12}}$$

and $v_n := \Delta_0$ in (7.7) then by integration by parts equation (7.7)

takes the form

$$(7.9) \quad -p(x_1)\Delta_0'(x_1-0)y_n(x_1) + p(0)\Delta_0'(0+0)y_n(0) - \\ - p(0)\Delta_0(0) \frac{a_{11}y_n(0) - c}{a_{12}} = \int_0^{x_1} f(t)\Delta_0(t)dt.$$

We have

$$\left\{ \begin{aligned} \Delta_0(t) &= \frac{\begin{vmatrix} b(t) & a(t) \\ b(x_1) & a(x_1) \end{vmatrix}}{D(0, x_1)} = \frac{D(t, x_1)}{D(0, x_1)}, \\ \Delta'_0(x_1-0) &= \frac{a(x_1)b'(x_1) - a'(x_1)b(x_1)}{D(0, x_1)} = \frac{1}{p(x_1)D(0, x_1)}, \\ \Delta'_0(0+0) &= \frac{b'(0)a(x_1) - a'(0)b(x_1)}{D(0, x_1)}, \\ a_{11}a(0) - a_{12}a'(0) &= 0. \end{aligned} \right.$$

The coefficient of $y_n(0)$ in (7.9) is

$$\begin{aligned} p(0)\Delta'_0(0+0) - p(0)\Delta_0(0) \frac{a_{11}}{a_{12}} &= \frac{p(0)}{a_{12}} [a_{12}\Delta'_0(0+0) - a_{11}\Delta_0(0)] = \\ &= \frac{p(0)}{a_{12}D(0, x_1)} \left[a_{12} \begin{vmatrix} b'(0) & a'(0) \\ b(x_1) & a(x_1) \end{vmatrix} - a_{11} \begin{vmatrix} b(0) & a(0) \\ b(x_1) & a(x_1) \end{vmatrix} \right] = \\ &= \frac{p(0)}{a_{12}D(0, x_1)} \begin{vmatrix} -B_1b & -B_1a \\ b(x_1) & a(x_1) \end{vmatrix} = -\frac{B_1(b)a(x_1)p(0)}{a_{12}D(0, x_1)} = \frac{a(x_1)}{a(0)D(0, x_1)}, \end{aligned}$$

since

$$\frac{p(0)}{a_{12}} = -\frac{1}{a(0)B_1(b)}$$

by (2.6).

The coefficient of $y_n(x_1)$ is

$$-p(x_1)\Delta'_0(x_1-0) = \frac{-1}{D(0, x_1)},$$

and the coefficient of c is

$$\frac{p(0)\Delta_0(0)}{a_{12}} = \frac{p(0)}{a_{12}} = -\frac{1}{a(0)B_1(b)}.$$

Hence equation (7.9) can be written as

$$(7.10) \quad \frac{a(x_1)}{a(0)D(0, x_1)} y_n(0) - \frac{y_n(x_1)}{D(0, x_1)} = \int_0^{x_1} f(t) \Delta_0(t) dt + \frac{c}{a(0)B_1(b)},$$

which is exactly equation (3.7) if we replace y_n by y .

If $b_{22} > 0$ put

$$y'(1) = \frac{d - b_{21}y(1)}{b_{22}}$$

and $y_n = \Delta_n$ in (7.7). Then by integration by parts equation (7.7) takes the form

$$(7.11) \quad p(x_{n-1})\Delta'_n(x_{n-1}+0)y_n(x_{n-1}) - p(1)\Delta'_n(1-0)y_n(1) + \\ + p(1)\Delta_n(1) \frac{d - b_{21}y_n(1)}{b_{22}} = \int_{x_{n-1}}^1 f(t)\Delta_n(t)dt.$$

We have

$$\left\{ \begin{aligned} \Delta_n(t) &= \frac{\begin{vmatrix} b(x_{n-1}) & a(x_{n-1}) \\ b(t) & a(t) \end{vmatrix}}{D(x_{n-1}, 1)} = \frac{D(x_{n-1}, t)}{D(x_{n-1}, 1)}, \\ \Delta'_n(1-0) &= \frac{b(x_{n-1})a'(1) - b'(1)a(x_{n-1})}{D(x_{n-1}, 1)}, \\ \Delta'_n(x_{n-1}+0) &= \frac{b(x_{n-1})a'(x_{n-1}) - b'(x_{n-1})a(x_{n-1})}{D(x_{n-1}, 1)} = \\ &= \frac{-1}{p(x_{n-1})D(x_{n-1}, 1)}, \\ b_{21}b(1) + b_{22}b'(1) &= 0. \end{aligned} \right.$$

The coefficient of $y_n(1)$ in (7.11) is

$$\begin{aligned}
 & -p(1)\Delta'_n(1-0) - p(1)\Delta_n(1) \frac{b_{21}}{b_{22}} = \\
 & = \frac{-p(1)}{b_{22}D(x_{n-1},1)} \left[b_{22} \begin{vmatrix} b(x_{n-1}) & a(x_{n-1}) \\ b'(1) & a'(1) \end{vmatrix} + b_{21} \begin{vmatrix} b(x_{n-1}) & a(x_{n-1}) \\ b(1) & a(1) \end{vmatrix} \right] = \\
 & = \frac{-p(1)}{b_{22}D(x_{n-1},1)} \begin{vmatrix} b(x_{n-1}) & a(x_{n-1}) \\ B_2(b) & B_2(a) \end{vmatrix} = - \frac{p(1)b(x_{n-1})B_2(a)}{b_{22}D(x_{n-1},1)} = \\
 & = \frac{b(x_{n-1})}{b(1)D(x_{n-1},1)},
 \end{aligned}$$

since

$$\frac{p(1)}{b_{22}} = \frac{-1}{b(1)B_2(a)}$$

by (2.7).

The coefficient of $y_n(x_{n-1})$ is

$$p(x_{n-1})\Delta'_n(x_{n-1}+0) = \frac{-1}{D(x_{n-1},1)},$$

and the coefficient of d is

$$\Delta_n(1) \frac{p(1)}{b_{22}} = \frac{p(1)}{b_{22}} = \frac{-1}{b(1)B_2(a)}.$$

Hence equation (7.11) can be written as

$$(7.12) \quad \frac{-y_n(x_{n-1})}{D(x_{n-1},1)} + \frac{b(x_{n-1})}{b(1)D(x_{n-1},1)} y_n(1) = \int_{x_{n-1}}^1 f(t)\Delta_n(t)dt + \frac{d}{b(1)B_2(a)},$$

which is exactly equation (3.9) if we replace y_n by y .

We have shown that Galerkin's method, with bas functions Δ_i , is equivalent to the determinant formulation of the problem. We note that the bas function Δ_i are optimal in the sense that they define

y_n exact on the grid $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$. It is interesting to notice that any function $w \in W^2$ which interpolate the exact solution y at x_i , $i = 0, \dots, n$, also satisfies equation (7.7) exactly when f is independent of y . We also note that the "natural" interpolation formula (5.17) imply that $\hat{y}$ in (5.17) is in the space V_n .

Consider the differential equation (7.1) with boundary conditions

$$\begin{cases} B_1 y = a_{11} y(0) = c, \\ B_2 y = b_{21} y(1) = d. \end{cases}$$

We assume that Green's function for this problem is given by (7.6), and the weak form by (7.4). With

$$S_n = W^1 \quad \text{and} \quad V_n = \{ v_n \in W_0^1 \mid v_n = \sum_{j=1}^{n-1} c_j \Delta_j \}$$

the Galerkin solution y_n is the element in W^1 which satisfies

$$(7.13) \quad \int_0^1 [-p(t)y_n'(t)v_n'(t) + q(t)y_n(t)v_n(t)] dt = \int_0^1 f(t)v_n(t) dt,$$

and the boundary conditions for all v_n in V_n . For $v_n := \Delta_i$, $i = 1, \dots, n-1$, in (7.13), this equation is equivalent to (7.7) and by integration by parts we get equation (7.8). Hence also in this case Galerkin's method is equivalent to the determinant formulation of the problem. We also note that any function $w \in W^1$ which interpolate the exact solution y at x_i , $i = 0, \dots, n$, also satisfies equation (7.13) exactly when f is independent of y .

8. DETERMINANT FORMULATION OF DISCRETE BOUNDARY VALUE PROBLEMS

We define a symmetric discrete boundary value problem by the equations

$$(8.1) \quad \begin{cases} d_0 y_0 + c_1 y_1 = f_0 \\ c_i y_{i-1} + d_i y_i + c_{i+1} y_{i+1} = f_i, & i = 1, \dots, n-1, \\ c_n y_{n-1} + d_n y_n = f_n, \end{cases}$$

where c_i, d_i, f_i are real numbers, and the f_i 's are allowed to be dependent on y . We also assume that $c_i \neq 0$, $i = 1, \dots, n$, since if one $c_i = 0$ the problem splits into two boundary value problems.

Let T be the symmetric tridiagonal matrix of the left hand side of (8.1), and $f = (f_0, f_1, \dots, f_n)^T$, $y = (y_0, y_1, \dots, y_n)^T$. Then equation (8.1) can be written as

$$(8.2) \quad Ty = f.$$

If T is nonsingular we get

$$(8.3) \quad y = T^{-1}f.$$

We call (8.3) the sum equation formulation of (8.1).

Theorem 8.1

If T is nonsingular then there exists two sequences $\{a_k\}_{k=0}^n$

and $\{b_k\}_{k=0}^n$ of real numbers such that $T^{-1} = G = (g_{i,j})_{i,j=0}^n$

where

$$(8.4) \quad g_{i,j} = \begin{cases} b_i a_j, & i \geq j, \\ a_i b_j, & i \leq j. \end{cases}$$

Proof: Define two vectors α and β by

$$(8.5) \quad T\beta = (1, 0, \dots, 0)^T, \quad T\alpha = (0, \dots, 0, 1)^T.$$

(β corresponds to the first column and α to the last column of the matrix G .) Since all $c_i \neq 0$ we must have $\alpha_0 \neq 0$ and $\beta_n \neq 0$

by (8.5). Now define the matrix G by

$$(8.6) \quad \varepsilon_{i,j} = \begin{cases} \beta_i \alpha_j / \alpha_0, & i \geq j, \\ \alpha_i \beta_j / \alpha_0, & i \leq j. \end{cases}$$

Hence we must show that $a_i = \alpha_i / \alpha_0$ and $b_i = \beta_i$. By (8.5) $TG = W$, where W is the diagonal matrix

$$W = \text{diag}(w_0, w_1, \dots, w_n),$$

where

$$(8.7) \quad \begin{cases} w_0 = d_0 a_0 b_0 + d_1 b_1 a_0, \\ w_i = c_i a_{i-1} b_i + d_i a_i b_i + c_{i+1} b_{i+1} a_i, \quad i = 1, \dots, n-1, \\ w_n = c_n a_{n-1} b_n + d_n a_n b_n. \end{cases}$$

Since $T\beta = (1, 0, \dots, 0)^T$ and $a_0 = 1$ we already have $w_0 = 1$.

By (8.5) we have

$$(8.8) \quad c_i b_{i-1} + d_i b_i + c_{i+1} b_{i+1} = 0, \quad i = 1, \dots, n-1,$$

$$(8.9) \quad c_i a_{i-1} + d_i a_i + c_{i+1} a_{i+1} = 0, \quad i = 1, \dots, n-1.$$

Hence we have

$$\begin{aligned} a_i [c_i b_{i-1} + d_i b_i + c_{i+1} b_{i+1}] - b_i [c_i a_{i-1} + d_i a_i + c_{i+1} a_{i+1}] &= \\ = c_i d(i-1, i) - c_{i+1} d(i, i+1) &= 0 \end{aligned}$$

or

$$c_{i+1} d(i, i+1) = c_i d(i-1, i), \quad i = 1, \dots, n-1,$$

where

$$d(i-1, i) = \begin{vmatrix} b_{i-1} & a_{i-1} \\ b_i & a_i \end{vmatrix}, \quad i = 1, \dots, n.$$

Thus $c_i d(i-1, i)$ is a constant function of i . By (8.5) the first row of $\beta_0 T\alpha - \alpha_0 T\beta$ is

$$\beta_0 [d_0 \alpha_0 + c_1 \alpha_1] - \alpha_0 [d_0 \beta_0 + c_1 \beta_1] = -\alpha_0$$

or

$$c_1[\beta_0 \alpha_1 - \beta_1 \alpha_0] = -\alpha_0,$$

and if we divide by α_0 we get

$$c_1 d(0,1) = -1.$$

Hence

$$(8.10) \quad c_i d(i-1,i) = -1, \quad i = 1, \dots, n.$$

By (8.8) we have

$$a_i[d_i b_i + c_{i+1} b_{i+1}] = -c_i b_{i-1} a_i, \quad i = 1, \dots, n-1.$$

Thus w_i can be written as

$$w_i = -c_i [b_{i-1} a_i - b_i a_{i-1}] = -c_i d(i-1,i) = 1,$$

for $i = 1, \dots, n-1$. Finally we show that $w_n = 1$. By the last row

of $T\beta = (1, 0, \dots, 0)^T$ we have

$$\alpha_n [c_n \beta_{n-1} + d_n \beta_n] = 0,$$

or if we divide by α_0 we get

$$a_n [c_n b_{n-1} + d_n b_n] = 0,$$

and by (8.10) we also have

$$c_n [b_{n-1} a_n - b_n a_{n-1}] = -1.$$

If we subtract these two equations we get

$$w_n = c_n a_{n-1} b_n + d_n a_n b_n = 1.$$

Remark 1.

Let g_0 be the first column of G and g_n the last. Consider Tg_0

and Tg_n . We have

$$(8.11) \quad \begin{cases} d_0 b_0 + c_1 b_1 = 1, \\ \vdots \\ c_{n-1} b_{n-2} + d_{n-1} b_{n-1} + c_n b_n = 0, \\ c_n b_{n-1} + d_n b_n = 0, \end{cases}$$

since $a_0 = 1$, and

$$(8.12) \begin{cases} d_0 a_0 + c_1 a_1 = 0, \\ c_1 a_0 + d_1 a_1 + c_2 a_2 = 0, \\ \vdots \\ c_{n-1} a_{n-2} + d_{n-1} a_{n-1} + c_n a_n = 0, \\ b_n (c_n a_{n-1} + d_n a_n) = 1, \end{cases}$$

since $b_n \neq 0$. Now since $a_0 = 1$ we can compute a_i , for $i = 1, \dots, n$, iteratively by (8.12) and also, by the last equation, b_n . With b_n known (8.11) defines an iterative method to compute b_i for $i = n-1, n-2, \dots, 0$. However since stability problem can occur it might be better to solve the linear system (8.11) first and after that the system

$$(8.13) \quad T a = (0, \dots, 0, 1/b_n)^T.$$

Then we also have the check for $a_0 = 1$.

Remark 2.

Consider the last equation in (8.12)

$$b_n (c_n a_{n-1} + d_n a_n) = 1.$$

Then $a_i = \alpha_i / \alpha_0$ gives

$$\frac{b_n}{\alpha_0} (c_n \alpha_{n-1} + d_n \alpha_n) = 1,$$

and by (8.5) we have $c_n \alpha_{n-1} + d_n \alpha_n = 1$. Hence $b_n = \beta_n = \alpha_0$.

See also Atkinson [1] section 6.4.

Remark 3.

By the proof of Theorem 3.3 we have

$$(8.14) \begin{cases} c_i = -1/(b_{i-1} a_i - b_i a_{i-1}), & i = 1, \dots, n, \\ d_i = \frac{b_{i-1} a_{i+1} - b_{i+1} a_{i-1}}{(b_{i-1} a_i - b_i a_{i-1})(b_i a_{i+1} - b_{i+1} a_i)}, & i = 0, \dots, n, \end{cases}$$

where $a_{-1} = b_{n+1} = 0$ and $b_{-1} \neq 0$, $a_{n+1} \neq 0$ can be chosen arbitrary.

We will now give an algorithm for the numerical solution of the two systems (8.11) and (8.12). The algorithm is based on the direct method for tridiagonal system, see Varga [26] p. 195.

Consider (8.11), following Varga we have

$$(i) \quad \begin{cases} w_n = \frac{c_n}{d_n}, \\ w_i = \frac{c_i}{d_i - c_{i+1}w_{i+1}}, \quad i = n-1, \dots, 1, \\ w_0 = \frac{1}{d_0 - c_1w_1}, \end{cases}$$

and

$$(ii) \quad \begin{cases} b_0 = w_0, \\ b_i = -w_i b_{i-1}, \quad i = 1, \dots, n. \end{cases}$$

By (8.14) we have

$$c_i = \frac{-1}{b_{i-1}a_i - b_i a_{i-1}}, \quad i = 1, \dots, n,$$

or

$$a_i = \frac{b_i}{b_{i-1}} a_{i-1} - \frac{1}{b_{i-1}c_i}, \quad i = 1, \dots, n,$$

and we also have $a_0 = 1$. But since,

$$b_i = -w_i b_{i-1}$$

we get

$$(iii) \quad \begin{cases} a_0 = 1 \\ a_i = -w_i a_{i-1} - \frac{1}{b_{i-1}c_i}, \quad i = 1, \dots, n. \end{cases}$$

Now let $a[0:n]$, $b[0:n]$, $c[0:n]$, $d[0:n]$ denote, respectively, the vectors $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, and $\{d_i\}$, then the solution of the two systems (8.11) and (8.12) is given by the following ALGOL program:


```

c[0] := 1; c[n+1] := 0;
for i := n step -1 until 0 do
    b[i] := c[i]/(d[i] - c[i+1] x b[i+1]);
a[0] := 1;
for i := 1 step 1 until n do
    begin
        a[i] := -(b[i] x a[i-1] + 1/(b[i-1] x c[i]));
        b[i] := - b[i] x b[i-1]
    end;

```

If we compare this algorithm with the iterative algorithm given in Remark 1 we see that the number of operations and overhead (two "for" statements) are exactly the same. Wilkinson [27] has shown that the direct method is extremely stable with respect to the growth of rounding errors when the tridiagonal matrix T is diagonally dominant (in practical problems this is almost always the case). Therefore the direct method of (i), (ii), and (iii) is numerically more desirable than the iterative method.

Equation (8.3) is called the sum equation formulation of (8.1) since, by Theorem 8.1, it can be written as

$$(8.15) \quad y_i = \sum_{k=0}^n g_{i,k} f_k, \quad i = 0, 1, \dots, n,$$

where

$$g_{i,k} = \begin{cases} b_i a_k, & i \geq k, \\ a_i b_k, & i \leq k. \end{cases}$$

We will now derive the determinant formulation of the sum equation (8.15).

Let

$$0 = n_0 < n_1 < \dots < n_{m-1} < n_m = n, \quad m < n,$$

be integers. By (8.15) we have

$$\left\{ \begin{aligned} y_{n_{i-1}} &= b_{n_{i-1}} \sum_{k=0}^{n_{i-1}-1} a_k^f k + \sum_{k=n_{i-1}+1}^{n_{i+1}-1} g_{n_{i-1},k}^f k + a_{n_{i-1}} \sum_{k=n_{i+1}}^n b_k^f k, \\ y_{n_i} &= b_{n_i} \sum_{k=0}^{n_i-1} a_k^f k + \sum_{k=n_{i-1}+1}^{n_{i+1}-1} g_{n_i,k}^f k + a_{n_i} \sum_{k=n_{i+1}}^n b_k^f k, \\ y_{n_{i+1}} &= b_{n_{i+1}} \sum_{k=0}^{n_{i+1}-1} a_k^f k + \sum_{k=n_{i-1}+1}^{n_{i+1}-1} g_{n_{i+1},k}^f k + a_{n_{i+1}} \sum_{k=n_{i+1}}^n b_k^f k, \end{aligned} \right.$$

for $i = 1, \dots, m-1$. As in the continuous case the determinant formulation of (8.15) is

$$(8.16) \quad \begin{vmatrix} b_{n_{i-1}} & a_{n_{i-1}} & y_{n_{i-1}} - \sum_{k=n_{i-1}+1}^{n_{i+1}-1} g_{n_{i-1},k}^f k \\ b_{n_i} & a_{n_i} & y_{n_i} - \sum_{k=n_{i-1}+1}^{n_{i+1}-1} g_{n_i,k}^f k \\ b_{n_{i+1}} & a_{n_{i+1}} & y_{n_{i+1}} - \sum_{k=n_{i-1}+1}^{n_{i+1}-1} g_{n_{i+1},k}^f k \end{vmatrix} = 0,$$

for $i = 1, \dots, m-1$. We also have

$$\left\{ \begin{aligned} y_0 &= \sum_{k=0}^{n_1-1} g_{0,k}^f k + a_0 \sum_{k=n_1}^n b_k^f k, \\ y_{n_1} &= \sum_{k=0}^{n_1-1} g_{n_1,k}^f k + a_{n_1} \sum_{k=n_1}^n b_k^f k, \end{aligned} \right.$$

which gives

$$(8.17) \quad \begin{vmatrix} a_0 & y_0 - \sum_{k=0}^{n_1-1} g_{0,k} f_k \\ a_{n_1} & y_{n_1} - \sum_{k=0}^{n_1-1} g_{n_1,k} f_k \end{vmatrix} = 0,$$

and at the other end point we have

$$\begin{cases} y_{n_{m-1}} = b_{n_{m-1}} \sum_{k=0}^{n_{m-1}} a_k f_k + \sum_{k=n_{m-1}+1}^n g_{n_{m-1},k} f_k, \\ y_n = b_n \sum_{k=0}^{n_{m-1}} a_k f_k + \sum_{k=n_{m-1}+1}^n g_{n,k} f_k, \end{cases}$$

which gives

$$(8.18) \quad \begin{vmatrix} b_{n_{m-1}} & y_{n_{m-1}} - \sum_{k=n_{m-1}+1}^n g_{n_{m-1},k} f_k \\ b_n & y_n - \sum_{k=n_{m-1}+1}^n g_{n,k} f_k \end{vmatrix} = 0.$$

We call (8.16), (8.17), and (8.18) the determinant formulation of the discrete boundary value problem (8.1).

Put

$$d(r,s) = \begin{vmatrix} b_r & a_r \\ b_s & a_s \end{vmatrix}, \quad 0 \leq r < s \leq n.$$

If $d(n_i, n_{i+1}) \neq 0$ for $i = 0, 1, \dots, m-1$, then (8.16) can be written as

$$\begin{aligned}
 (8.19) \quad & \frac{-y_{n_{i-1}}}{d(n_{i-1}, n_i)} + \frac{d(n_{i-1}, n_{i+1})}{d(n_{i-1}, n_i) d(n_i, n_{i+1})} y_{n_i} + \frac{-y_{n_{i+1}}}{d(n_i, n_{i+1})} = \\
 & = \sum_{k=n_{i-1}+1}^{n_{i+1}-1} \Delta_{i,k} f_k, \quad i = 1, \dots, m-1,
 \end{aligned}$$

where

$$(8.20) \quad \Delta_{i,k} = \begin{cases} \frac{d(n_{i-1}, k)}{d(n_{i-1}, n_i)}, & k = n_{i-1}, \dots, n_i, \\ \frac{d(k, n_{i+1})}{d(n_i, n_{i+1})}, & k = n_i, \dots, n_{i+1}. \end{cases}$$

In the same way (8.17) can be written as

$$(8.21) \quad \frac{a_{n_1}}{a_0 d(0, n_1)} y_0 - \frac{y_{n_1}}{d(0, n_1)} = \sum_{k=0}^{n_1-1} \Delta_{0,k} f_k, \quad \text{if } a_0 \neq 0,$$

where

$$(8.22) \quad \Delta_{0,k} = \frac{d(k, n_1)}{d(0, n_1)}, \quad k = 0, \dots, n_1,$$

and (8.18) takes the form

$$(8.23) \quad \frac{-y_{n_{m-1}}}{d(n_{m-1}, n)} + \frac{b_{n_{m-1}}}{b_n d(n_{m-1}, n)} y_n = \sum_{k=n_{m-1}+1}^n \Delta_{n,k} f_k, \quad \text{if } b_n \neq 0,$$

where

$$(8.24) \quad \Delta_{n,k} = \frac{d(n_{m-1}, k)}{d(n_{m-1}, n)}, \quad k = n_{m-1}, \dots, n.$$

With $a_{n_{-1}} = b_{n_{m+1}} = 0$ and $b_{n_{-1}} \neq 0, a_{n_{m+1}} \neq 0$ equations (8.21)

and (8.23) can be written as

$$(8.21') \quad \frac{d(n_{-1}, n_1)}{d(n_{-1}, 0)d(0, n_1)} y_0 + \frac{-y_{n_1}}{d(0, n_1)} = \sum_{k=0}^{n_1-1} \Delta_{0,k} f_k,$$

$$(8.23') \quad \frac{-y_{n_{m-1}}}{d(n_{m-1}, n)} + \frac{d(n_{m-1}, n_{m+1})}{d(n_{m-1}, n)d(n, n_{m+1})} y_n = \sum_{k=n_{m-1}}^n \Delta_{n,k} f_k.$$

The equations (8.19), (8.21), and (8.23) define a system for the unknowns y_{n_i} , $i = 0, 1, \dots, m$. The matrix of this system is

$$(8.25) \quad T_m = \begin{bmatrix} \frac{a_{n_1}}{a_0 d(0, n_1)} & \frac{-1}{d(0, n_1)} & & & & & 0 \\ & \frac{d(0, n_2)}{d(0, n_1)d(n_1, n_2)} & \frac{-1}{d(n_1, n_2)} & & & & \\ & & & \ddots & & & \\ & & & & \frac{d(n_{m-2}, n)}{d(n_{m-2}, n_{m-1})d(n_{m-1}, n)} & \frac{-1}{d(n_{m-1}, n)} & \\ & & & & & \frac{b_{n_{m-1}}}{b_n d(n_{m-1}, n)} & \\ 0 & & & & \frac{-1}{d(n_{m-1}, n)} & & \end{bmatrix}$$

By Theorem 3.3 the matrix T_m is nonsingular if and only if $a_0, b_n \neq 0$, $d(n_i, n_{i+1}) \neq 0$ for $i = 0, 1, \dots, m-1$. We will now give two theorems which assure that $d(k, k+r) \neq 0$ for $0 \leq k < k+r \leq n$, where k and r are integers. The following theorem corresponds to Theorem 3.1 in the continuous case.

Theorem 8.2

Assume that $c_i > 0$, $i = 1, \dots, n$, and that T is negative definite. Then $d(k, k+r) \neq 0$ for $0 \leq k < k+r \leq n$.

Proof: Consider the system

$$\begin{cases} Bb_k + Aa_k = 0 \\ Bb_{k+r} + Aa_{k+r} = 0. \end{cases}$$

If $d(k, k+r) = 0$ this system has a nontrivial solution (A, B) and

$$y_j = Aa_j + Bb_j, \quad j = k, \dots, k+r,$$

is a nontrivial solution of the homogeneous difference equation

$$\begin{cases} c_i y_{i-1} + d_i y_i + c_{i+1} y_{i+1} = 0, & i = k+1, \dots, k+r-1, \\ y_k = y_{k+r} = 0, & c_{k+1}, c_{k+r} > 0. \end{cases}$$

But since all the principal minors of T are positive this system can only have the trivial solution $y_i = 0, i = k, \dots, k+r$.

Remark.

Since T is negative definite all eigenvalues λ of $\det(T + \lambda I) = 0$ are greater than zero (compare with Theorem 3.7).

We will now give a condition which ensure that T is negative definite.

Proposition 8.3

Let A be a Stieltjes matrix. Then A is positive definite.

Proof: (See Ortega [16] p. 109.)

Proposition 8.4

Let A be a strictly or irreducibly diagonally dominant $n \times n$ matrix and assume that $a_{ij} \leq 0, i \neq j$, and $a_{ii} > 0, i = 1, \dots, n$. Then A is an M-matrix.

Proof: (See Ortega [16] p. 110.)

Consider the matrix $-T$ and assume that

$$(8.26) \quad \begin{cases} c_i > 0, & i = 1, \dots, n, \\ c_i + c_{i+1} \leq -d_i, & i = 1, \dots, n-1, \\ c_1 \leq -d_0, \quad c_n \leq -d_n & \text{with strict inequality for } c_1 \text{ or } c_n. \end{cases}$$

Then T is irreducibly diagonally dominant and by Proposition 8.4

$-T$ is an M-matrix. Since $-T$ is symmetric, it is a Stieltjes matrix and by Proposition 8.3 $-T$ is positive definite. Hence T is negative definite if (8.26) holds.

If T is known to be nonsingular then we have the following theorem.

Theorem 8.5

Suppose that $c_i > 0$, $i = 1, \dots, n$, $c_i + c_{i+1} \leq -d_i$ for $i = 1, \dots, n-1$, and T is nonsingular. Then we have $d(k, k+r) \neq 0$ for $0 \leq k < k+r \leq n$.

Proof: Assume that $d(k, k+r) = 0$, $0 \leq k < k+r \leq n$. Consider row $k+i+1$, $i = 1, \dots, r-1$, of $b_k Ta - a_k Tb$. By (8.11) and (8.12) we have

$$\begin{aligned} b_k [c_{k+i} a_{k+i-1} + d_{k+i} a_{k+i} + c_{k+i+1} a_{k+i+1}] &= 0, \\ a_k [c_{k+i} b_{k+i-1} + d_{k+i} b_{k+i} + c_{k+i+1} b_{k+i+1}] &= 0, \end{aligned}$$

which give

$$(8.27) \quad \begin{cases} c_{k+i} d(k, k+i-1) + d_{k+i} d(k, k+i) + c_{k+i+1} d(k, k+i+1) = 0, \\ \text{for } i = 1, \dots, r-1, \quad d(k, k) = d(k, k+r) = 0. \end{cases}$$

By the maximum principle (see Ortega [16] p. 97) we must have $d(k, k+i) = 0$ for $i = 0, \dots, r$. But since T is nonsingular we must have $d(k, k+1) \neq 0$. Hence we must have $d(k, k+r) \neq 0$.

To derive the determinant formulation given by (8.19), (8.21), and (8.23) we have to solve two linear systems (8.11) and (8.12) and then forming the determinants $\Delta_{i,k}$. However we know that

$$\Delta_{1, n_1-1} = 0 \quad \text{and} \quad \Delta_{1, n_1} = 1 \quad \text{and also that} \quad \Delta_{1,k} \text{ satisfies (8.27)}$$

for suitable values on k and i . Hence $\Delta_{i,k}$ can be obtained by solving smaller tridiagonal systems. Let

$$(8.28) \quad T_{i,i+1} = \begin{bmatrix} d_{n_i+1} & c_{n_i+2} & & 0 \\ c_{n_i+2} & d_{n_i+2} & c_{n_i+3} & \\ & \ddots & \ddots & \ddots \\ & & c_{n_{i+1}-2} & d_{n_{i+1}-2} & c_{n_{i+1}-1} \\ 0 & & & c_{n_{i+1}-1} & d_{n_{i+1}-1} \end{bmatrix},$$

for $i = 0, 1, \dots, m-1$.

Theorem 8.6

Define the vectors Δ_i^+ and Δ_i^- by

$$(8.29) \quad \begin{cases} \Delta_i^+ = (\Delta_{i+1, n_i+1}, \dots, \Delta_{i+1, n_{i+1}-1})^T, \\ \Delta_i^- = (\Delta_{i, n_i+1}, \dots, \Delta_{i, n_{i+1}-1})^T. \end{cases}$$

Then we have

$$(8.30) \quad \begin{cases} T_{i,i+1} \Delta_i^+ = (0, \dots, 0, -c_{n_{i+1}})^T \\ T_{i,i+1} \Delta_i^- = (-c_{n_i+1}, 0, \dots, 0)^T, \end{cases}$$

for $i = 0, \dots, m-1$, and Δ_i^+, Δ_i^- are uniquely defined by (8.30) if T satisfies the hypothesis of Theorem 8.5.

Proof: By (8.11) and (8.12) we have

$$a_{n_i} (c_k b_{k-1} + d_k b_k + c_{k+1} b_{k+1}) = 0,$$

$$b_{n_i} (c_k a_{k-1} + d_k a_k + c_{k+1} a_{k+1}) = 0,$$

for $k = n_i+1, \dots, n_{i+1}-1$, $i = 0, 1, \dots, m-1$. By subtracting these two equations we get

$$c_k d(n_i, k-1) + d_k d(n_i, k) + c_{k+1} d(n_i, k+1) = 0$$

and if we divide by $d(n_i, n_{i+1})$ we get

$$c_k \frac{d(n_i, k-1)}{d(n_i, n_{i+1})} + d_k \frac{d(n_i, k)}{d(n_i, n_{i+1})} + c_{k+1} \frac{d(n_i, k+1)}{d(n_i, n_{i+1})} = 0.$$

But since $d(n_i, n_i)/d(n_i, n_{i+1}) = 0$, $d(n_i, n_{i+1})/d(n_i, n_{i+1}) = 1$ we get

$$\left\{ \begin{array}{l} d_{n_i+1} \frac{d(n_i, n_{i+1})}{d(n_i, n_{i+1})} + c_{n_i+2} \frac{d(n_i, n_{i+2})}{d(n_i, n_{i+1})} = 0, \\ c_k \frac{d(n_i, k-1)}{d(n_i, n_{i+1})} + d_k \frac{d(n_i, k)}{d(n_i, n_{i+1})} + c_{k+1} \frac{d(n_i, k+1)}{d(n_i, n_{i+1})} = 0, \\ \text{for } k = n_i+2, \dots, n_{i+1}-2, \\ c_{n_{i+1}-1} \frac{d(n_i, n_{i+1}-2)}{d(n_i, n_{i+1})} + d_{n_{i+1}-1} \frac{d(n_i, n_{i+1}-1)}{d(n_i, n_{i+1})} = -c_{n_{i+1}}, \end{array} \right.$$

for $i = 0, \dots, m-1$, which is the first equation in (8.30). It is clear that the matrix of this system is nonsingular if T satisfies the hypothesis of Theorem 8.5. In the same way we can show that the second equation in (8.30) holds.

The elements of the matrix T_m defined by (8.25) are given by the following theorem.

Theorem 8.7

We have

$$(8.31) \quad \frac{-1}{d(n_i, n_{i+1})} = c_{n_i+1} \Delta_{i+1, n_{i+1}} = c_{n_i+1} \Delta_{i, n_{i+1}-1}, \quad i = 0, \dots, m-1,$$

$$(8.32) \quad \frac{d(n_{i-1}, n_{i+1})}{d(n_{i-1}, n_i) d(n_i, n_{i+1})} = c_{n_i} \Delta_{i, n_i-1} + d_{n_i} \Delta_{i, n_i} + c_{n_i+1} \Delta_{i, n_i+1},$$

for $i = 1, \dots, m-1$ ($\Delta_{i, n_i} = 1$),

$$(8.33) \quad \frac{a_{n_1}}{a_0 d(0, n_1)} = d_0 + c_1 \Delta_{0, 1},$$

$$(8.34) \quad \frac{b_{n_{m-1}}}{b_n d(n_{m-1}, n)} = c_n \Delta_{m, n-1} + d_n.$$

Proof: By (8.14) we have $c_i = -1/d(i-1, i)$, which give

$$c_{n_i+1} \Delta_{i+1, n_i+1} = \frac{-1}{d(n_i, n_{i+1})} \frac{d(n_i, n_{i+1})}{d(n_i, n_{i+1})} = \frac{-1}{d(n_i, n_{i+1})}$$

and

$$c_{n_{i+1}} \Delta_{i, n_{i+1}-1} = \frac{-1}{d(n_{i+1}-1, n_{i+1})} \frac{d(n_{i+1}-1, n_{i+1})}{d(n_i, n_{i+1})} = \frac{-1}{d(n_i, n_{i+1})}.$$

Hence we have shown (8.31). We have

$$\begin{vmatrix} d(n_{i-1}, n_i) & d(n_i, n_{i+1}) \\ d(n_{i-1}, n_{i-1}) & d(n_{i-1}, n_{i+1}) \end{vmatrix} = d(n_{i-1}, n_{i+1}) d(n_{i-1}, n_i).$$

Hence

$$(8.35) \quad \frac{d(n_{i-1}, n_{i+1})}{d(n_i, n_{i+1})} - \frac{d(n_{i-1}, n_{i-1})}{d(n_{i-1}, n_i)} = \frac{d(n_{i-1}, n_{i+1})}{d(n_{i-1}, n_i) d(n_i, n_{i+1})} d(n_{i-1}, n_i).$$

By (8.11) and (8.12) we also have

$$(8.36) \quad c_{n_i} \frac{d(n_{i-1}, n_{i+1})}{d(n_i, n_{i+1})} + d_{n_i} \frac{d(n_i, n_{i+1})}{d(n_i, n_{i+1})} + c_{n_{i+1}} \frac{d(n_{i+1}, n_{i+1})}{d(n_i, n_{i+1})} = 0.$$

If we multiply (8.35) by $c_{n_i} = -1/d(n_{i-1}, n_i)$ and eliminate

$$c_{n_i} \frac{d(n_{i-1}, n_{i+1})}{d(n_i, n_{i+1})}$$

by (8.36), we get (8.32).

Since

$$\begin{aligned} d_0 + c_1 \Delta_{0,1} &= \frac{a_1}{a_0 d(0,1)} + \frac{-1}{d(0,1)} \frac{d(1, n_1)}{d(0, n_1)} = \\ &= \frac{1}{a_0 d(0,1) d(0, n_1)} [a_1 d(0, n_1) - a_0 d(1, n_1)] = \\ &= \frac{1}{a_0 d(0,1) d(0, n_1)} \begin{vmatrix} b_0 a_1 - b_1 a_0 & a_0 a_1 - a_1 a_0 \\ b_{n_1} & a_{n_1} \end{vmatrix} = \frac{a_{n_1}}{a_0 d(0, n_1)} \end{aligned}$$

and

$$\begin{aligned}
c_n \Delta_{m,n-1} + d_n &= \frac{-1}{d(n-1,n)} \frac{d(n_{m-1},n-1)}{d(n_{m-1},n)} + \frac{b_{n-1}}{b_n d(n-1,n)} = \\
&= \frac{1}{b_n d(n-1,n) d(n_{m-1},n)} [b_{n-1} d(n_{m-1},n) - b_n d(n_{m-1},n-1)] = \\
&= \frac{1}{b_n d(n-1,n) d(n_{m-1},n)} \begin{vmatrix} b_{n_{m-1}} & a_{n_{m-1}} \\ b_{n-1} b_n - b_n b_{n-1} & b_{n-1} a_n - b_n a_{n-1} \end{vmatrix} = \\
&= \frac{b_{n_{m-1}}}{b_n d(n_{m-1},n)}
\end{aligned}$$

(8.33) and (8.34) hold.

By Theorem 8.6 $\Delta_{i,k}$, $i = 0, 1, \dots, m$, are defined by $2m$ tridiagonal systems which all can be solved in parallel.

We will now consider nonsymmetric discrete boundary value problems of the form

$$(8.37) \quad \begin{cases} d_0 y_0 + e_1 y_1 = f_0, \\ c_i y_{i-1} + d_i y_i + e_{i+1} y_{i+1} = f_i, \quad i = 1, \dots, n-1, \\ c_n y_{n-1} + d_n y_n = f_n, \end{cases}$$

where c_i, e_i, f_i are real numbers and the f_i 's may depend on y .

We also assume that $c_i e_i \neq 0$ for $i = 1, \dots, n$.

Let T denote the tridiagonal matrix of the left hand side of (8.37)

and let $f = (f_0, f_1, \dots, f_n)^T$, $y = (y_0, y_1, \dots, y_n)^T$. Then equation

(8.37) can be written as

$$(8.38) \quad Ty = f.$$

Following Ortega [16] p. 113 we can transform this problem into a symmetric one if $c_i e_i > 0$ for $i = 1, \dots, n$. Put

$$(8.39) \quad D = \text{diag}\left(1, \sqrt{\frac{e_1}{c_1}}, \sqrt{\frac{e_1 e_2}{c_1 c_2}}, \dots, \sqrt{\frac{e_1 \dots e_n}{c_1 \dots c_n}}\right).$$

Then (8.38) can be written as

$$DTD^{-1}Dy = Df,$$

where

$$(8.40) \quad DTD^{-1} = \begin{pmatrix} d_0 & \gamma_1 & & & 0 \\ \gamma_1 & d_1 & \gamma_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_{n-1} & d_{n-1} & \gamma_n \\ 0 & & & \gamma_n & d_n \end{pmatrix}$$

and $\gamma_i = \sqrt{c_i e_i}$, $i = 1, \dots, n$.

We will follow another course here.

Theorem 8.8

The matrix T of (8.1) is nonsingular if and only if there exists two vectors $a = (a_0, \dots, a_n)^T$ and $b = (b_0, \dots, b_n)^T$ such that

$$(8.41) \quad \begin{cases} Tb = (1, 0, \dots, 0)^T \\ Ta = (0, \dots, 0, 1)^T, \end{cases}$$

with $a_0 b_n \neq 0$ and $d(i-1, i) \neq 0$ for $i = 1, \dots, n$.

Proof: Assume that (8.41) holds and that $a_0 b_n \neq 0$, $d(i-1, i) \neq 0$ for $i = 1, \dots, n$. Define the matrix $\hat{G} = (\hat{g}_{i,j})_{i,j=0}^n$ by

$$\hat{g}_{i,j} = \begin{cases} b_i a_j, & i \geq j, \\ a_i b_j, & i \leq j. \end{cases}$$

By (8.41) we have

$$(8.42) \quad \widehat{TG} = W = \text{diag}(w_0, w_1, \dots, w_n),$$

where

$$(8.43) \quad \begin{cases} w_0 = d_0 a_0 b_0 + e_1 b_1 a_0, \\ w_i = c_i a_{i-1} b_i + d_i a_i b_i + e_{i+1} b_{i+1} a_i, \quad i = 1, \dots, n-1, \\ w_n = c_n a_{n-1} b_n + d_n a_n b_n. \end{cases}$$

By (8.41) we have

$$w_0 = a_0 [d_0 b_0 + e_1 b_1] = a_0 \neq 0,$$

$$w_n = b_n [c_n a_{n-1} + d_n a_n] = b_n \neq 0,$$

and

$$d_i b_i + e_{i+1} b_{i+1} = -c_i b_{i-1}$$

$$\Rightarrow d_i a_i b_i + e_{i+1} b_{i+1} a_i = -c_i b_{i-1} a_i.$$

By (8.43) we get

$$(8.44) \quad w_i = c_i a_{i-1} b_i - c_i b_{i-1} a_i = -c_i d(i-1, i) \neq 0, \quad i = 1, \dots, n-1.$$

Hence W is nonsingular and by Theorem 3.3 $\widehat{G}$ is nonsingular. Thus

T is nonsingular and

$$(8.45) \quad T^{-1} = \widehat{GW}^{-1}.$$

Assume that T is nonsingular. Then we can solve (8.41) for a

and b . Since $c_i e_i \neq 0$ we must have $a_0 b_n \neq 0$. By (8.41) we have

$$\begin{cases} c_i b_{i-1} + d_i b_i + e_{i+1} b_{i+1} = 0, & i = 1, \dots, n-1, \\ c_i a_{i-1} + d_i a_i + e_{i+1} a_{i+1} = 0, & i = 1, \dots, n-1. \end{cases}$$

We get

$$\begin{aligned} a_i [c_i b_{i-1} + d_i b_i + e_{i+1} b_{i+1}] - b_i [c_i a_{i-1} + d_i a_i + e_{i+1} a_{i+1}] &= \\ &= c_i d(i-1, i) - e_{i+1} d(i, i+1) = 0 \end{aligned}$$

or

$$(8.46) \quad d(i, i+1) = \frac{c_i}{e_{i+1}} d(i-1, i), \quad i = 1, \dots, n-1.$$

By (8.41) we also have

$$\begin{cases} d_0 b_0 + e_1 b_1 = 1, \\ d_0 a_0 + e_1 a_1 = 0, \end{cases}$$

$$\Rightarrow a_0 [d_0 b_0 + e_1 b_1] - b_0 [d_0 a_0 + e_1 a_1] = a_0$$

$$\Rightarrow -e_1 [b_0 a_1 - b_1 a_0] = a_0$$

$$\Rightarrow d(0,1) = -a_0/e_1.$$

Hence we have

$$(8.47) \quad d(i-1,i) = -\frac{c_{i-1}c_{i-2}\dots c_1 c_0}{e_i e_{i-1}\dots e_1} a_0 \neq 0, \quad i = 1, \dots, n,$$

where $c_0 = 1$.

Corollary 1

If T is nonsingular then we have

$$(8.48) \quad T^{-1} = G = (g_{i,j})_{i,j=0}^n$$

where

$$(8.49) \quad g_{i,j} = \begin{cases} b_i a_j / w_j, & i \geq j, \\ a_i b_j / w_j, & i \leq j, \end{cases}$$

and

$$(8.50) \quad Ta = (0, \dots, 0, 1)^T,$$

$$(8.51) \quad Tb = (1, 0, \dots, 0)^T,$$

$$(8.52) \quad \begin{cases} w_0 = a_0, & w_n = b_n, \\ w_i = -c_i d(i-1,i) = \frac{c_1 \dots c_0}{e_i \dots e_0} a_0, & i = 0, \dots, n, \end{cases}$$

where $c_0 = e_0 = 1$, $d(-1,0) = a_0$.

Proof: The Corollary follows immediately from (8.44), (8.45), (8.47), and (8.41). We also have

$$a_n [c_n b_{n-1} + d_n b_n] - b_n [c_n a_{n-1} + d_n a_n] = -b_n$$

$$\Rightarrow c_n [b_{n-1} a_n - b_n a_{n-1}] = -b_n = -w_n,$$

or

$$w_n = - \frac{c_n \dots c_0}{e_n \dots e_0} a_0 = b_n.$$

Remark.

If $c_i = e_i$, $i = 1, \dots, n$, then $a_0 = b_n$ as we indicated in Remark 2 of Theorem 8.1.

Corollary 2

We have

$$(8.53) \quad c_i = \frac{-w_i}{d(i-1, i)}, \quad i = 1, \dots, n,$$

$$(8.54) \quad e_i = \frac{-w_{i-1}}{d(i-1, i)}, \quad i = 1, \dots, n,$$

$$(8.55) \quad \begin{cases} d_0 = \frac{w_0 a_1}{a_0 d(0, 1)}, \\ d_i = \frac{w_i d(i-1, i+1)}{d(i-1, i) d(i, i+1)}, \quad i = 1, \dots, n-1, \\ d_n = \frac{w_n b_{n-1}}{b_n d(n-1, n)}. \end{cases}$$

Proof: We have $T = W\hat{T}$, where

$$(8.56) \quad \hat{T} = \begin{bmatrix} \frac{a_1}{a_0 d(0, 1)} & \frac{-1}{d(0, 1)} & & \\ \frac{-1}{d(0, 1)} & \frac{d(0, 2)}{d(0, 1) d(1, 2)} & \frac{-1}{d(1, 2)} & \\ & \frac{-1}{d(n-2, n-1)} & \frac{d(n-2, n)}{d(n-2, n-1) d(n-1, n)} & \frac{-1}{d(n-1, n)} \\ & & \frac{-1}{d(n-1, n)} & \frac{b_{n-1}}{b_n d(n-1, n)} \end{bmatrix}$$

We say that T satisfies the maximum principle if the elements in T or $-T$ satisfy the inequalities

$$(8.57) \quad c_i > 0, \quad e_i > 0, \quad \text{for } i = 1, \dots, n,$$

and

$$(8.58) \quad c_i + e_{i+1} \leq -d_i, \quad \text{for } i = 1, \dots, n-1.$$

Theorem 8.9

Let T satisfies the maximum principle. Then $\hat{T} = W^{-1}T$ also satisfies the maximum principle.

Proof: The elements of W are of one sign.

Hence if T satisfies the maximum principle, we have by Theorem 8.5 $d(k, k+r) \neq 0$, $0 \leq k < k+r \leq n$. The determinant formulation of the symmetric problem

$$(8.59) \quad \hat{T}y = W^{-1}f$$

is then given by (8.19), (8.21), and (8.23), where we have put

$$f_i := f_i/w_i, \quad i = 0, 1, \dots, n.$$

Another way to obtain a symmetric problem is to put

$$(8.60) \quad \begin{cases} w_0 = 1, \\ w_i = \frac{c_i}{e_i} w_{i-1}, \quad i = 1, \dots, n, \end{cases}$$

and $W = \text{diag}(w_0, w_1, \dots, w_n)$. Then we have the symmetric boundary value problem

$$(8.61) \quad (W^{-1}T)y = W^{-1}f,$$

where

$$(8.62) \quad W^{-1}T = \begin{bmatrix} \delta_0 & \gamma_1 & & & \\ \gamma_1 & \delta_1 & \gamma_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_n & \delta_n & \end{bmatrix}$$

$$(8.63) \quad \begin{cases} \gamma_0 = 1, & \gamma_i = \frac{e_i \dots e_1}{c_{i-1} \dots c_1}, \quad i = 2, \dots, n, \\ \delta_0 = d_0, & \delta_i = \frac{e_i \dots e_1}{c_i \dots c_1} d_i, \quad i = 1, \dots, n. \end{cases}$$

APPENDIX

We will define Newton-like methods for the numerical solution of the boundary value problem (2.1). Newton's method for the integral equation (2.5) is

$$(A1) \quad \begin{cases} u(x) - \int_0^1 G(x,t) f'_y(t, y(t)) u(t) dt = \\ = - \left[y(x) - \int_0^1 G(x,t) f(t, y(t)) dt - \frac{cb(x)}{B_1(b)} - \frac{da(x)}{B_2(a)} \right], \\ y(x) := y(x) + u(x), \quad 0 \leq x \leq 1. \end{cases}$$

Let n, m , and r be positive integers and $n = rm$. Then m and n define two grids $G_m = \{\frac{j}{m} \mid j = 0, \dots, m\}$ and $G_n = \{\frac{i}{n} \mid i = 0, \dots, n\}$ on the interval $[0, 1]$. To construct Newton-like methods for the procedure (A1) we approximate the left hand side of (A1) on the grid G_m and the right hand side on the grid G_n . We will only consider the trapezoidal rule here. We get

$$(A2) \quad \begin{aligned} u(x) - \sum_{j=0}^m \omega_j G(x, \frac{j}{m}) f'_y(\frac{j}{m}, y(\frac{j}{m})) u(\frac{j}{m}) = \\ = - \left[y(x) - \sum_{k=0}^n w_k G(x, \frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})) - \frac{cb(x)}{B_1(b)} - \frac{da(x)}{B_2(a)} \right], \end{aligned}$$

for $0 \leq x \leq 1$, where ω_j and w_k are the weights of the trapezoidal rule (for simplicity we use u and y in both (A1) and (A2)).

The determinant formulation of (A2) on the grid G_m is

$$(A3) \quad \begin{aligned} & \frac{-u(\frac{i-1}{m})}{D(\frac{i-1}{m}, \frac{i}{m})} + \frac{D(\frac{i-1}{m}, \frac{i+1}{m})}{D(\frac{i-1}{m}, \frac{i}{m}) D(\frac{i}{m}, \frac{i+1}{m})} u(\frac{i}{m}) + \frac{-u(\frac{i+1}{m})}{D(\frac{i}{m}, \frac{i+1}{m})} - \frac{1}{m} f'_y(\frac{i}{m}, y(\frac{i}{m})) u(\frac{i}{m}) = \\ & = - \left[\frac{-y(\frac{i-1}{m})}{D(\frac{i-1}{m}, \frac{i}{m})} + \frac{D(\frac{i-1}{m}, \frac{i+1}{m})}{D(\frac{i-1}{m}, \frac{i}{m}) D(\frac{i}{m}, \frac{i+1}{m})} y(\frac{i}{m}) + \frac{-y(\frac{i+1}{m})}{D(\frac{i}{m}, \frac{i+1}{m})} - \right. \\ & \quad \left. - \sum_{k=r(i-1)+1}^{r(i+1)-1} \frac{1}{n} a_i(\frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})) \right], \end{aligned}$$

for $i = 1, \dots, m-1$. If $a_{12} > 0$ we have

$$\begin{aligned}
 (A4) \quad & \frac{a(\frac{1}{m})}{a(0)D(0,\frac{1}{m})} u(0) + \frac{-u(\frac{1}{m})}{D(0,\frac{1}{m})} - \frac{1}{2m} f'_y(0, y(0)) u(0) = \\
 & = - \left[\frac{a(\frac{1}{m})}{a(0)D(0,\frac{1}{m})} y(0) + \frac{-y(\frac{1}{m})}{D(0,\frac{1}{m})} - \sum_{k=0}^{r-1} w_k \Delta_0(\frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})) - \right. \\
 & \quad \left. - \frac{c}{a(0)B_1(b)} \right].
 \end{aligned}$$

If $b_{22} > 0$ we have

$$\begin{aligned}
 (A5) \quad & \frac{-u(\frac{m-1}{m})}{D(\frac{m-1}{m}, 1)} + \frac{b(\frac{m-1}{m})}{b(1)D(\frac{m-1}{m}, 1)} u(1) - \frac{1}{2m} f'_y(1, y(1)) u(1) = \\
 & = - \left[\frac{-y(\frac{m-1}{m})}{D(\frac{m-1}{m}, 1)} + \frac{b(\frac{m-1}{m})}{b(1)D(\frac{m-1}{m}, 1)} y(1) - \sum_{k=r(m-1)+1}^n w_k \Delta_m(\frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})) - \right. \\
 & \quad \left. - \frac{d}{b(1)B_2(a)} \right].
 \end{aligned}$$

Now (A3), (A4), and (A5) define u on the grid G_m . To compute intermediate values of u (on the grid G_n) consider the determinant formulation of equation (A2) at the points $\frac{i-1}{m} < \frac{j}{n} < \frac{i}{m}$. We get

$$\begin{aligned}
 (A6) \quad & u(\frac{j}{n}) = \Delta_{i-1}(\frac{j}{n}) [y(\frac{i-1}{m}) + u(\frac{i-1}{m})] + \Delta_i(\frac{j}{n}) [y(\frac{i}{m}) + u(\frac{i}{m})] - y(\frac{j}{n}) + \\
 & + D(\frac{i-1}{m}, \frac{i}{m}) \left[\Delta_{i-1}(\frac{j}{n}) \sum_{k=r(i-1)+1}^j \frac{1}{n} \Delta_i(\frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})) + \right. \\
 & \quad \left. + \Delta_i(\frac{j}{n}) \sum_{k=j+1}^{ri-1} \frac{1}{n} \Delta_{i-1}(\frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})) \right],
 \end{aligned}$$

for $j = r(i-1)+1, \dots, ri-1$, $i = 1, \dots, m$.

Now (A3), (A4), (A5) followed by equation (A6) define a Newton-like method for the sum equation

$$(A7) \quad y(\frac{i}{n}) = \sum_{k=0}^n w_k G(\frac{i}{n}, \frac{k}{n}) f(\frac{k}{n}, y(\frac{k}{n})) - \frac{cb(\frac{i}{n})}{B_1(b)} - \frac{da(\frac{i}{n})}{B_2(a)}, \quad i = 0, \dots, n.$$

For more details see Aulin [2], [3], [5], [6].

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